Scattering & Blow-up Phenomena for Semi-Linear Wave Equations

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0 Notation

We will consider equations on a space-time domain, with coordinates $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. For a given point $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$, we denote the negative light-cone emanating from the point (x_0,t_0) by

$$C(x_0,t_0) = \{(x,t)\colon t \le t_0, |x - x_0| \le t_0 - t\}$$

and we denote by $\mathcal{M}(x_0,t_0) = \partial C(x_0,t_0)$ the mantle (boundary) of the negative light-cone. We denote truncated cones/mantles by

$$C_s^t(x_0,t_0) = C(x_0,t_0) \cap \{\mathbb{R}^n \times [s,t]\}$$

$$\mathcal{M}_s^t(x_0,t_0) = \mathcal{M}(x_0,t_0) \cap \{\mathbb{R}^n \times [s,t]\}$$

and we denote the cross sections of the cone by

$$D_{(x_0,t_0)}(s) = C(x_0,t_0) \cap \{\mathbb{R}^n \times \{s\}\}\,.$$

For the forward light cone we use the notation

$$\Gamma(x_0,t_0) = \{(x,t)\colon t \ge t_0, |x - x_0| \le t - t_0\}\,.$$

Whenever the given point is the origin, we omit the label, writing $C(0,0) = C$ etc. We denote by $d\sigma$ the volume measure on the unit sphere; by $d\sigma_r$ the volume measure on the sphere of radius r; and by $d\omega = \sqrt{2}d\sigma_t dt$ the volume measure on the mantle $\mathcal{M}_0^t(x_0,t_0)$. Throughout the report, for a function of space-time $w\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ we make the abuse of notation $w(t) := w(\cdot,t)$ whenever the space dependence is implicit.

1 The linear wave equation

The main source for the linear theory of this section was the book of Shatah and Struwe [12]. Indeed, the entire section, with the exception of the energy estimates, is directly following chapter 4 of this book, and all proofs are taken from there.

1.1 Representation formulas

The Cauchy problem for the linear wave equation is concerned with finding, and quantifying, a solution $u\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ to

$$\begin{cases} \Box u = f = f(x,t), \\ u(0) = u_0; \quad \partial_t u(0) = u_1, \end{cases} \tag{1}$$

where $\Box = \partial_t^2 - \Delta_x$ is the d'Alembert operator, for a given u_0, u_1. A general solution to (1) may be constructed via the fundamental solution $\mathcal{R} = \mathcal{R}(x,t)$, which solves

$$\begin{cases} \Box\mathcal{R} = 0, \\ \mathcal{R}(0) = 0; \quad \partial_t\mathcal{R}(0) = \delta_0, \end{cases} \tag{2}$$

in the distributional sense. Applying the Fourier transform to (2) with respect to the space coordinate gives

$$\begin{cases} (\partial_{tt})\widehat{\mathcal{R}}(\xi,t) + |\xi|^2\widehat{\mathcal{R}}(\xi,t) = 0, \\ \widehat{\mathcal{R}}(\xi,0) = 0; \quad \partial_t\widehat{\mathcal{R}}(\xi,0) = 1, \end{cases}$$

and solving this ODE in t gives $\widehat{\mathcal{R}}(\xi,t) = \frac{\sin(|\xi|t)}{|\xi|}$. Thus the fundamental solution is given by

$$\mathcal{R}(t) = \mathcal{F}^{-1}\left(\frac{\sin(|\cdot|t)}{|\cdot|}\right), \tag{3}$$

where $\mathcal{F}^{-1}$ is the inverse spacial Fourier transform. For each t, there is a spherical symmetry of $\mathcal{R}(t)$ arising from the rotational invariance of Δ.

Using the fundamental solution, we can check that a solution $\underline{u}$ to the homogeneous system

$$\begin{cases} \Box\underline{u} = 0, \\ \underline{u}(0) = u_0; \quad \partial_t\underline{u}(0) = u_1 \end{cases}, \tag{4}$$

is given by

$$\underline{u}(t) = \mathcal{R}(t) * u_1 + \partial_t\mathcal{R}(t) * u_0, \tag{5}$$

where $*$ denotes convolution. Then by the Duhamel principle, the general solution to (1) is given by

$$u(t) = \underline{u}(t) - \int_0^t \mathcal{R}(t-\tau) * f(\tau)d\tau. \tag{6}$$

This method gives a solution to (1) in the distributional sense even for rough data, as we may view u_0, u_1, f as tempered distributions. Mostly though, we will consider smooth initial data in which case u will be a solution in the classical sense. For the nonlinear equations later on, it will be helpful to write (6) as

$$u = \underline{u} + \mathcal{L}(\Box(u)), \tag{7}$$

which holds for any tempered distribution u, where $\mathcal{L}$ is the Duhamel operator defined by

$$(\mathcal{L}w)(t) = \int_0^t \mathcal{R}(t-\tau) * w(\tau)d\tau. \tag{8}$$

When $n = 1$, we have $\mathcal{R}(x,t) = \mathcal{F}^{-1}\left(\frac{\sin(|\cdot|t)}{|\cdot|}\right)(x) = \frac{1}{2}\mathbb{1}_{[-t,t]}(x)$, and from (5) we recover d'Alembert's formula

$$\underline{u}(x,t) = \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}(u_0(x+t) + u_0(x-t)). \tag{9}$$

Then by the Duhamel principle,

$$u(x,t) = \underline{u}(x,t) + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,\tau)dyd\tau. \tag{10}$$

For higher dimensions it is more work to compute the fundamental solution. This is usually presented, for n odd, by the *method of spherical means*, where the rotational invariance of Δ is used to reduce to an ODE. For n even, the solution is then obtained by the *method of descent*. We give the derivations here for the cases $n = 3$ and $n = 2$.

For a given point $x_0 \in \mathbb{R}^3$, define a spherically averaged function

$$\tilde{w}(r,t) = \frac{1}{4\pi}\int_{|y|=1} w(x_0 + ry, t)d\sigma(y). \tag{11}$$

Expresing Δ in spherical polar coordinates, (4) becomes

$$\underline{u}_{tt} - \underline{u}_{rr} - \frac{2}{r}\underline{u}_r - \Delta_{S^2}\underline{u} = 0,$$

where Δ_{S^2} is the Laplace–Beltrami operator on the sphere. Taking spherical averages then gives, by Stokes' theorem,

$$\tilde{\underline{u}}_{tt} - \tilde{\underline{u}}_{rr} - \frac{2}{r}\tilde{\underline{u}}_r = 0.$$

Making the substitution $v(r,t) = r\tilde{\underline{u}}(r,t)$, this becomes the 1-dimensional wave equation $v_{tt} - v_{rr} = 0$, and so by (9) we have

$$v(r,t) = \frac{1}{2}\left(\int_{r-t}^{r+t} s\tilde{u}_1(s)ds\right) + \frac{1}{2}((r+t)\tilde{u}_0(r+t) + (r-t)\tilde{u}_0(r-t)).$$

To obtain $\underline{u}$, note that $\underline{u}(x_0,t) = \tilde{\underline{u}}(0,t) = \partial_r\big|_{r=0} v(r,t)$ and so

$$\begin{aligned}\underline{u}(x_0,t) &= \frac{1}{2}\big((r+t)\tilde{u}_1(r+t) - (r-t)\tilde{u}_1(r-t) + \tilde{u}_0(r+t) + \tilde{u}_0(r-t) \\ &\qquad + (r+t)\partial_r\tilde{u}_0(r+t) + (r-t)\partial_r\tilde{u}_0(r-t)\big)\Big|_{r=0} \\ &= t\tilde{u}_1(t) + \tilde{u}_0(t)t(\partial_r\tilde{u}_0)(t) \\ &= \frac{t}{4\pi}\int_{y=1} u_1(x_0+ty)d\sigma(y) + \frac{1}{4\pi}\int_{|y|=1}(u_0(x_0+ty) + t(\nabla u_0\cdot\eta)(x_0+ty))d\sigma(y),\end{aligned} \tag{12}$$

where we noted that $\tilde{u}_0(r) = \tilde{u}_0(-r)$, $\tilde{u}_1(r) = \tilde{u}_1(-r)$ and hence $(\partial_r\tilde{u}_0)(r) = -(\partial_r\tilde{u}_0)(-r))$. Here η denotes the outward unit normal. Making the change of variables $z = x + ty$, we then rewrite the solution as

$$\underline{u}(x,t) = \frac{1}{4\pi t}\int_{\partial B_t(x)} u_1 d\sigma_t + \frac{1}{4\pi t^2}\int_{\partial B_t(x)}(u_0 + t(\nabla u_0\cdot\eta))d\sigma_t. \tag{13}$$

We thus see that the fundamental solution for $n = 3$ is

$$\mathcal{R}(t,x) = \frac{1}{4\pi t}\delta(t - |x|), \tag{14}$$

where $\delta(t-|x|)$ is the distribution which acts by integration over the mantle of the forward lightcone, $\Gamma = (x,t)\colon t = |x|$. The Duhamel principle then gives the solution as

$$u(x,t) = \underline{u}(x,t) + \frac{1}{4\sqrt{2}\pi}\int_{\mathcal{M}_0^t(x,t)} t^{-1} f(y,t)d\omega(y,t), \tag{15}$$

and reparametrising gives the solution as

$$u(x,t) = \underline{u}(x,t) + \frac{1}{4\pi}\int_{B_t(x)} t^{-1} f(x+y, t-|y|)dy. \tag{16}$$

Now, for the case $n=2$, given any solution $\underline{u}$ of (4), we arrive at a solution $\hat{u}$ in dimension $n=3$ by defining $\underline{\hat{u}}(x^1,x^2,x^3,t) = \underline{u}(x^1,x^2,t)$. From (13) we then have

$$\underline{\hat{u}}(x,t) = \frac{1}{4\pi t}\int_{\partial B_t(x)} \hat{u}_1(z)d\sigma_t(z) + \frac{1}{4\pi t^2}\int_{\partial B_t(x)} (\hat{u}_0(z) + t(\nabla \hat{u}_0 \cdot \eta))(z)d\sigma_t(z),$$

where $\hat{u}_0(x^1,x^2,x^3) = u_0(x^1,x^2)$, $\hat{u}_1(x^1,x^2,x^3) = u_1(x^1,x^2)$. Since everything here is independent of the x^3-coordinate, for the sphere in $z = (x^1,x^2,x^3)$ space, we parametrise both hemispheres $\{x^3 > 0\}$, $\{x^3 < 0\}$, by the $y = (x^1,x^2)$ coordinates. Then $d\sigma_t(z) = \frac{t}{\sqrt{t^2-|x-y|^2}}dy$ and we arrive at

$$\underline{u}(x,t) = \frac{2}{4\pi}\int_{B_t(x)} \frac{u_1(y)}{\sqrt{t^2-|x-y|^2}}dy + \frac{2}{4\pi t}\int_{B_t(x)} \frac{u_0 + t(\nabla u_0\cdot\eta)(y)}{\sqrt{t^2-|x-y|^2}}dy. \tag{17}$$

The fundamental solution when $n=2$ is thus

$$\mathcal{R}(t,x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbb{1}_{B_t(x)}. \tag{18}$$

We refer to [12] for the derivation of the representation formulas in higher dimensions. In odd dimension $n = 2m+3$, the fundamental solution is given by

$$\mathcal{R}(t,x) = A_n\left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left(\frac{1}{t}\delta(|x|-t)\right), \tag{19}$$

where $A_n = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)(n-4)\cdots 3\cdot 1}$. In even dimensions $n = 2m+2$, the fundamental solution is given by

$$\mathcal{R}(t,x) = A_n\left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}}\left(\frac{1}{\sqrt{t^2-|x|^2}}\mathbb{1}_{B_t(x)}\right). \tag{20}$$

Note the following observations following from the representation formulas given by (6)

Theorem 1.1 (Huygen's principle)**.** *In $n=1$ space dimension and in all the even space dimensions, the solution u at a point (x_0,t_0) depends only on the values of f over the backward light-cone $C_0^{t_0}(x_0,t_0)$, and only on the values of the initial data u_0, u_1 over the base of this cone $B_{t_0}(x_0)$.*
In odd space dimensions greater than or equal to three, however, the solution u at a point (x_0,t_0) depends only on the values of f over the mantle of the backward light-cone $\mathcal{M}_0^{t_0}(x_0,t_0)$, and only on the values of the initial data u_0, u_1 over the base of this mantle $\partial B_{t_0}(x_0)$. Thus in these dimensions, information propagates along a sharp wavefront.

Corollary 1.2 (Finite speed of propagation)**.** *Given initial data u_0, u_1 supported in a ball $B_R(0)$, the homogeneous solution $\underline{u}$ takes support*

$$\mathrm{supp}(\underline{u}) \subset \{(x,t); |x| \leq R + t\}.$$

i.e. information propagates at a finite speed.

1.2 Dispersive and energy estimates

From the representation formulas, we can read off pointwise estimates for solutions to the linear wave equation. These are referred to as *decay* or *dispersive* as they describe the decay of a solution. From d'Alembert's formula (9), the first estimate is immediate.

Lemma 1.3 (Dispersive estimate in 1D)**.** *When $n = 1$, solutions $\underline{u}$ to the homogeneous wave equation* (4) *satisfy*

$$\|\underline{u}(t)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}\left(\|u_0\|_{L^\infty(\mathbb{R})} + \|u_1\|_{L^1(\mathbb{R})}\right). \tag{21}$$

For the case $n = 3$, we note that at time $t = 1$ it holds

$$\begin{aligned} |\underline{u}(x,1)| &\leq \frac{1}{4\pi}\int_{\partial B_1(x)} |u_1| d\sigma + \frac{1}{4\pi}\int_{\partial B_1(x)} (|u_0| + |\nabla u_0|) d\sigma \\ &\leq \frac{1}{4\pi}\int_{B_1(x)} (|u_1| + |\nabla u_1|) dy + \frac{1}{4\pi}\int_{B_1(x)} (|u_0| + 2|\nabla u_0| + |\nabla^2 u_0|) dy \\ &\leq C\left(\int_{\mathbb{R}^3} |\nabla u_1| dy + \int_{\mathbb{R}^3} |\nabla^2 u_0| dy\right), \end{aligned} \tag{22}$$

where we used the trace embedding $W^{1,1} \hookrightarrow L^1$ and the result, by the Hölder and Sobolev inequalities, that for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \int_{B_1(x)} |\varphi| dy &\leq \left(\int_{B_1(x)} dy\right)^{\frac{1}{3}} \left(\int_{B_1(x)} |\varphi|^{\frac{3}{2}} dy\right)^{\frac{2}{3}} \\ &\leq \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |\varphi|^{\frac{3}{2}} dy\right)^{\frac{2}{3}} \\ &\leq \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |\nabla \varphi| dy\right). \end{aligned} \tag{23}$$

Noting that, for any $\lambda > 0$, $\underline{u}^\lambda(x,t) = \underline{u}(\lambda x, \lambda t)$ also solves (4) with $\underline{u}^\lambda(x,0) = u_0(\lambda x)$, $\partial_t \underline{u}^\lambda(x,0) = \lambda u_1(\lambda x)$, and as the homogeneous Sobolev norms scale as $\|\varphi(\lambda(\cdot))\|_{\dot{W}^{1,1}(\mathbb{R}^3)} = \lambda^{-2}\|\varphi(\cdot)\|_{\dot{W}^{1,1}(\mathbb{R}^3)}$ and $\|\varphi(\lambda(\cdot))\|_{\dot{W}^{2,1}(\mathbb{R}^3)} = \lambda^{-1}\|\varphi(\cdot)\|_{\dot{W}^{1,1}(\mathbb{R}^3)}$, we arrive at the following result.

Lemma 1.4 (Dispersive estimate in 3D)**.** *When $n = 3$, solutions $\underline{u}$ to the homogeneous wave equation* (4) *satisfy*

$$\|\underline{u}(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t}\left(\|u_0\|_{\dot{W}^{2,1}(\mathbb{R}^3)} + \|u_1\|_{\dot{W}^{1,1}(\mathbb{R}^3)}\right). \tag{24}$$

For the case $n=2$, we estimate the representation formula via the following calculation, using parts to trade regularity. Let $\varphi \in \mathcal{C}^\infty(B_1(0))$, $\varphi \equiv 1$ for $\frac{1}{2} \leq |y| \leq 1$, $\varphi \equiv 0$ near 0, then

$$\begin{aligned}\int_{|y|\leq 1} \frac{g(y)}{\sqrt{1-|y|^2}}dy &= \int_{\frac{1}{2}\leq|y|\leq 1} \frac{g(y)}{\sqrt{1-|y|^2}}dy + \int_{0\leq|y|\leq\frac{1}{2}} \frac{g(y)}{\sqrt{1-|y|^2}}dy \\ &\leq \int_{|y|\leq 1} \varphi\frac{g(y)}{\sqrt{1-|y|^2}}dy + C\|g\|_{L^1(B_1(0))} \\ &= \int_{|y|\leq 1} \varphi g(y)\frac{-y}{|y|^2}\cdot\nabla(\sqrt{1-|y|^2})dy + C\|g\|_{L^1(B_1(0))},\end{aligned} \tag{25}$$

where we noted that $\nabla\cdot(\frac{y}{|y|^2}) = \delta_0$ in 3 dimensions.

Now, just taking $\int_{|y|\leq 1}|\nabla g(y)|\varphi\frac{\sqrt{1-|y|^2}}{|y|}dy \leq \int_{|y|\leq 1}|\nabla g(y)|dy$, we would obtain from (17) and (23)

$$|\underline{u}(x,1)| \leq C(\|u_1\|_{\dot{W}^{1,1}(\mathbb{R}^2)} + \|u_0\|_{\dot{W}^{2,1}(\mathbb{R}^2)}). \tag{26}$$

But, since $\|\varphi(\lambda(\cdot))\|_{\dot{W}^{2,1}(\mathbb{R}^2)} = \|\varphi(\cdot)\|_{\dot{W}^{2,1}(\mathbb{R}^2)}$, this is not sharp enough to show any decay of the solution.

Instead, we note that $\varphi\frac{\sqrt{1-|y|^2}}{|y|} \in \mathcal{C}^{1/2} \hookrightarrow \dot{B}^{1/2}_{\infty,\infty}$, where $\dot{B}$ denotes the homogeneous Besov space. Since $(\dot{B}^{1/2}_{\infty,\infty})^* \cong \dot{B}^{-1/2}_{1,1}$ it follows that

$$\int_{|y|\leq 1}|\nabla g(y)|\varphi\frac{\sqrt{1-|y|^2}}{|y|}dy \leq C\|\nabla g\|_{\dot{B}^{-1/2}_{1,1}(\mathbb{R}^2)} \leq C\|g\|_{\dot{B}^{1/2}_{1,1}(\mathbb{R}^2)}. \tag{27}$$

This extra fractional derivative is enough to obtain decay of the solution after rescaling. As $\|\varphi(\lambda(\cdot))\|_{\dot{B}^{3/2}_{1,1}(\mathbb{R}^2)} = \lambda^{-1/2}\|\varphi(\cdot)\|_{\dot{B}^{3/2}_{1,1}(\mathbb{R}^2)}$, applying (17) and (23) we arrive at

Lemma 1.5 (Dispersive estimate in 2D)**.** *When n=2, solutions $\underline{u}$ to the homogeneous wave equation* (4) *satisfy*

$$\|\underline{u}\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{t^{1/2}}\left(\|u_0\|_{\dot{B}^{3/2}_{1,1}(\mathbb{R}^2)} + \|u_1\|_{\dot{B}^{1/2}_{1,1}(\mathbb{R}^2)}\right). \tag{28}$$

Similar computations for the higher dimensional formulas give

Lemma 1.6 (Dispersive estimates in higher dimensions)**.** *In odd dimensions $n=2m+3$, solutions $\underline{u}$ to the homogeneous wave equation* (4) *satisfy*

$$\|\underline{u}(t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{t^{\frac{n-1}{2}}}\left(\|u_0\|_{\dot{W}^{\frac{n+1}{2},1}(\mathbb{R}^n)} + \|u_1\|_{\dot{W}^{\frac{n-1}{2},1}(\mathbb{R}^n)}\right), \tag{29}$$

and in even dimensions $n=2m+2$

$$\|\underline{u}(t,x)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{t^{\frac{n-1}{2}}}\left(\|u_0\|_{\dot{B}^{\frac{n+1}{2}}_{1,1}(\mathbb{R}^n)} + \|u_1\|_{\dot{B}^{\frac{n-1}{2}}_{1,1}(\mathbb{R}^n)}\right). \tag{30}$$

The wave equation also admits L^2-bounds, referred to as energy estimates. Assuming compactly supported initial data u_0, u_1, by finite speed of propagation it follows that $\underline{u}(t)$ is compactly supported for all $t>0$. Multiplying through (4) by $\underline{u}_t$ and integrating by parts then gives

$$\underline{E}'(t) = \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\underline{u}_t|^2 + |\nabla\underline{u}|^2dx\right) = 0, \tag{31}$$

and so it follows

$$\|\underline{u}_t(t)\|^2_{L^2(\mathbb{R}^n)} + \|\underline{u}(t)\|_{\dot{H}^1(\mathbb{R}^n)} = \|u_1\|^2_{L^2(\mathbb{R}^n)} + \|u_0\|^2_{\dot{H}^1(\mathbb{R}^n)}. \tag{32}$$

Going even further, taking the Fourier transform of (4) with respect to the space coordinate, and multiplying through by $|\xi|^{2s}\hat{\underline{u}}_t$ gives

$$\begin{aligned} 0 &= |\xi|^{2s}\hat{\underline{u}}_t(\hat{\underline{u}}_{tt} + |\xi|^2\hat{\underline{u}}) \\ &= \frac{d}{dt}\left((|\xi|^s\hat{\underline{u}}_t)^2 + (|\xi|^{s+1}\hat{\underline{u}})^2\right), \end{aligned} \tag{33}$$

and thus

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\int_{\mathbb{R}^n}(|\xi|^s\hat{\underline{u}}_t)^2 d\xi + \int_{\mathbb{R}^n}(|\xi|^{s+1}\hat{\underline{u}})^2 d\xi\right) \\ &= \frac{d}{dt}\left(\|\hat{\underline{u}}_t\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\hat{\underline{u}}\|^2_{\dot{H}^{s+1}(\mathbb{R}^n)}\right), \end{aligned} \tag{34}$$

by Plancherel's theorem. We then arrive at

Theorem 1.7 (Energy estimates)**.** *A solution $\underline{u}$ to the homogeneous wave equation* (4) *satisfies*

$$\|\underline{u}_t(t)\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\underline{u}(t)\|^2_{\dot{H}^{s+1}(\mathbb{R}^n)} = \|u_1\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|u_0\|^2_{\dot{H}^{s+1}(\mathbb{R}^n)} \tag{35}$$

for any $s \in \mathbb{R}$.

Let us draw attention to the following two consequences of the energy estimates.

Corollary 1.8. *Finite energy solutions to the linear wave equation are unique.*

Corollary 1.9. *The Duhamel operator* (8)*, when viewed as a map $\mathcal{L}\colon L^\infty([0,T];\dot{H}^s(\mathbb{R}^n)) \to L^\infty([0,T];\dot{H}^s(\mathbb{R}^n))$, is bounded, with*

$$\|\mathcal{L}(w)\|_{L^\infty([0,T];\dot{H}^s(\mathbb{R}^n))} \le T\|w\|_{L^\infty([0,T];\dot{H}^s(\mathbb{R}^n))}. \tag{36}$$

1.3 Strichartz estimates

Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\eta = 1$ on $B_1(0)$ and $\eta = 0$ outside $B_2(0)$. Denote $\beta(\xi) = \eta(\xi) - \eta(\frac{\xi}{2})$ and $\varphi_k(\xi) = \beta(2^{-k}\xi)$. For $k \in \mathbb{Z}$, the φ_k form a partition for the punctured frequency space $\mathbb{R}^n \setminus \{0\}$, where each φ_k is supported on an annulus

$$\operatorname{supp}(\varphi_k) \subset \{\xi\colon 2^k \le |\xi| \le 2^{k+2}\}.$$

We define the Littlewood-Payley projections by

$$P_k(v) = v * \check{\varphi}_k$$

where ˇ denotes the inverse spacial Fourier transform, so that $\widehat{P_k(v)} = \hat{v} \cdot \varphi_k$. The homogeneous Besov norm is given then by

$$\|v\|_{\dot{B}^s_{q,r}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} \left(2^{sk} \|P_k(v)\|_{L^q(\mathbb{R}^n)} \right)^r \right)^{\frac{1}{r}}.$$

We will see now that the Besov norm lends itself to interpolated decay estimates, called Strichartz estimates. Firstly, let $\underline{u}$ solve (4) with $u_0 = 0$, so that $\underline{u}(t) = \mathcal{R}(t) * u_1$. Thus

$$\|\underline{u}(t)\|_{\dot{B}^s_{q,r}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} \left(2^{sk} \|a_k(t) * u_1\|_{L^q(\mathbb{R}^n)} \right)^r \right)^{\frac{1}{r}}, \tag{37}$$

where $a_k(t) = P_k\left(\mathcal{R}(t)\right) = \mathcal{R}(t) * \check{\varphi}_k$.

We then calculate

$$\begin{aligned} a_k(x,t) &= \left(\mathcal{F}^{-1}\left(\frac{\sin|\cdot|t}{|\cdot|} \right) * \check{\varphi}_k \right)(x) \\ &= \mathcal{F}^{-1}\left(\frac{\sin|\cdot|t}{|\cdot|} \beta\left(2^{-k}(\cdot)\right) \right)(x) \\ &= \int \frac{\sin|\xi|t}{|\xi|} \beta(2^{-k}\xi) e^{i\xi x} d\xi \\ &= 2^{k(n-1)} \int \frac{\sin|\eta|2^k t}{|\eta|} \beta(\eta) e^{i\eta 2^k x} d\eta \\ &= 2^{k(n-1)} \mathcal{F}^{-1}\left(\frac{\sin|\cdot|2^k t}{|\cdot|} \right)(2^k x) \\ &= 2^{k(n-1)} \left(\mathcal{R}(2^k t) * \check{\beta} \right)(2^k x), \end{aligned} \tag{38}$$

and

$$\hat{a}_k(\xi,t) = \frac{\sin|\xi|t}{|\xi|} \cdot \varphi_k(\xi). \tag{39}$$

From the dispersive estimates Lemma 1.6 and the embedding $\dot{B}_{1,1}^{\frac{n-1}{2}} \hookrightarrow \dot{W}^{\frac{n-1}{2},1}$ we have

$$\|\mathcal{R}(2^k t) * \check{\beta}\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{(2^k t)^{\frac{n-1}{2}}} \|\check{\beta}\|_{\dot{B}_{1,1}^{\frac{n-1}{2}}(\mathbb{R}^n)} \tag{40}$$

and so (38) and (39) give

$$\|a_k(t)\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{t^{\frac{n-1}{2}}} 2^{k\left(\frac{n-1}{2}\right)}, \tag{41}$$

$$\|\hat{a}_k(t)\|_{L^\infty(\mathbb{R}^n)} \le 2^{-k}, \tag{42}$$

which then implies

$$\|a_k(t) * u_1\|_{L^\infty} \le \|a_k(t)\|_{L^\infty} \|u_1\|_{L^1} \le \frac{C}{t^{\frac{n-1}{2}}} 2^{k\left(\frac{n-1}{2}\right)} \|u_1\|_{L^1}, \tag{43}$$

$$\|a_k(t) * u_1\|_{L^2} = \|\hat{a}_k(t) \cdot \hat{u}_1\|_{L^2} \le 2^{-k} \|u_1\|_{L^2}. \tag{44}$$

In other words, we have an operator bounded between two distinct pairs

$$a_k(t) * (\cdot) \colon L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \qquad \text{with } \|a_k(t) * (\cdot)\|_{L^1 \to L^\infty} \leq \frac{C}{t^{\frac{n-1}{2}}} 2^{k(\frac{n-1}{2})}, \tag{45}$$

$$a_k(t) * (\cdot) \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \qquad \text{with } \|a_k(t) * (\cdot)\|_{L^2 \to L^2} \leq 2^{-k}. \tag{46}$$

Applying the Riesz-Thorin interpolation theorem then gives that we have a bounded operator

$$a_k(t) * (\cdot) \colon L^p \to L^q$$

for all $\theta \in (0,1)$, where

$$\begin{aligned} \frac{1}{p} &= (1-\theta) \cdot \frac{1}{1} + \theta \cdot \frac{1}{2} = 1 - \frac{\theta}{2}, \\ \frac{1}{q} &= (1-\theta) \cdot \frac{1}{\infty} + \theta \cdot \frac{1}{2} = \frac{\theta}{2}, \end{aligned}$$

which satisfies

$$\|a_k(t) * (\cdot)\|_{L^p \to L^q} \leq \left(\frac{C}{t^{\frac{n-1}{2}}} 2^{k(\frac{n-1}{2})} \right)^{1-\theta} 2^{-k\theta} = \frac{C}{t^{(1-\frac{2}{q})(\frac{n-1}{2})}} 2^{k(\frac{n-1}{2} - \frac{n+1}{q})}. \tag{47}$$

We apply this to interpolate between Besov norms

$$\begin{aligned} \|\underline{u}(t)\|_{\dot{B}^s_{q,r}(\mathbb{R}^n)} &= \left(\sum_{k=-\infty}^{\infty} \left(2^{sk} \|a_k(t) * u_0\|_{L^q(\mathbb{R}^n)} \right)^r \right)^{\frac{1}{r}} \\ &\leq \left(\sum_{k=-\infty}^{\infty} \left(2^{sk} \left(\frac{C}{t^{(1-\frac{2}{q})(\frac{n-1}{2})}} 2^{k(\frac{n-1}{2} - \frac{n+1}{q})} \|u_0\|_{L^p(\mathbb{R}^n)} \right) \right)^r \right)^{\frac{1}{r}} \\ &= \frac{C}{t^{(1-\frac{2}{q})(\frac{n-1}{2})}} \left(\sum_{k=-\infty}^{\infty} \left(2^{k(s + \frac{n-1}{2} - \frac{n+1}{q})} \|u_0\|_{L^p(\mathbb{R}^n)} \right)^r \right)^{\frac{1}{r}} \\ &= \frac{C}{t^{(1-\frac{2}{q})(\frac{n-1}{2})}} \|u_0\|_{\dot{B}^{s'}_{p,r}(\mathbb{R}^n)}, \end{aligned} \tag{48}$$

where $s' = s + (\frac{n-1}{2} - \frac{n+1}{q})$.

Next, let $\underline{u}$ solve (4) with $u_1 = 0$, so that $\underline{u}(t) = \partial_t \mathcal{R}(t) * u_0$, where $\partial_t \mathcal{R}(t) = \partial_t \mathcal{F}^{-1} \left(\frac{\sin(|\xi| t)}{|\xi|} \right) = \mathcal{F}^{-1} \left(\cos(|\xi| t) \right)$. Let $b_k(t) = P_k(\partial_t \mathcal{R}(t)) = \partial_t \mathcal{R}(t) * \check{\varphi}_k$ be the Littlewood-Payley projections, so that

$$\|\underline{u}(t)\|_{\dot{B}^s_{q,r}} = \left(\sum_{k=-\infty}^{\infty} \left(2^{sk} \|b_k(t) * u_0\|_{L^q} \right)^r \right)^{\frac{1}{r}}. \tag{49}$$

This time,

$$\begin{aligned} b_k(x,t) &= \left(\mathcal{F}^{-1} \left(\cos(|\cdot| t) \right) * \varphi_k \right)(x) \\ &= 2^{nk} \mathcal{F}^{-1} \left(\cos(|\cdot| 2^k t) \beta(\cdot) \right)(2^k x) \\ &= 2^{nk} \left(\partial_t \mathcal{R}(2^k t) * \check{\beta} \right)(2^k x) \end{aligned} \tag{50}$$

and by the decay estimate $\|\partial_t \mathcal{R}(2^k t) * \check{\beta}\|_{L^\infty} \leq \frac{C}{(2^k t)^{\frac{n-1}{2}}} \|\check{\beta}\|_{\dot{B}^{\frac{n+1}{2}}_{1,1}}$, we have

$$\begin{aligned} \|b_k(t) * u_0\|_{L^\infty} \leq \|b_k(t)\|_{L^\infty} \|u_0\|_{L^1} \leq \frac{C}{t^{\frac{n-1}{2}}} 2^{j(\frac{n+1}{2})} \|u_0\|_{L^1}, \\ \|b_k(t) * u_0\|_{L^\infty} \leq \|\hat{b}_k(t)\|_{L^\infty} \|u_0\|_{L^2} \leq \|u_0\|_{L^2}. \end{aligned} \tag{51}$$

Applying Riesz-Thorin then gives

$$\|b_k(t) * (\cdot)\|_{L^p \to L^q} \leq \left(\frac{C}{t^{\frac{n-1}{2}}} 2^{k(\frac{n+1}{2})} \right)^{1-\frac{2}{q}} = \frac{C}{t^{(1-\frac{2}{q})(\frac{n-1}{2})}} 2^{k(\frac{n+1}{2} - \frac{n+1}{q})}, \tag{52}$$

when $\frac{1}{p} + \frac{1}{q} = 1$, and we deduce

$$\|\underline{u}(t)\|_{\dot{B}^s_{q,r}(\mathbb{R}^n)} \leq \frac{C}{t^{(1-\frac{2}{q})(\frac{n-1}{2})}} \|u_0\|_{\dot{B}^{s''}_{p,r}(\mathbb{R}^n)}, \tag{53}$$

where $s'' = s + \frac{n+1}{2} - \frac{n+1}{q} = s' + 1$.

Summarizing this, we have proved the following.

Lemma 1.10 (p-q estimate)**.** *Solutions to the homogeneous wave equation* (4) *satisfy*

$$\|\underline{u}(t)\|_{\dot{B}^s_{q,r}(\mathbb{R}^n)} \leq \frac{C}{t^{(1-\frac{2}{q})(\frac{n-2}{2})}} \left(\|u_1\|_{\dot{B}^{s'}_{p,r}(\mathbb{R}^n)} + \|u_0\|_{\dot{B}^{s'+1}_{p,r}(\mathbb{R}^n)} \right), \tag{54}$$

where $\frac{1}{p} + \frac{1}{q} = 1$*, and* $s' = s + \frac{n-1}{2} - \frac{n+1}{q}$*.*

Note that we did not interpolate with the energy estimates to obtain this result, we only used the decay of the solution. The underlying trick was the change of variables in step (38), where the high frequencies of the fundamental solution are estimated by the frequency $|\xi| \sim 1$ at a later time, and low frequencies are estimated by the frequency $|\xi| \sim 1$ at an earlier time.

As presented in [12], it is easier to deduce estimates for the function

$$\mathcal{U}(t) = \mathcal{F}^{-1} \left(\frac{e^{i|\xi|t}}{|\xi|} \right) \tag{55}$$

as we will see in the proof of the next lemma, taking advantage of the composition rule $e^{i|\xi|t} \cdot e^{i|\xi|s} = e^{i|\xi|(t+s)}$.

We note that since

$$\begin{aligned} \frac{\sin(|\xi|t)}{|\xi|} &= \frac{1}{2} \left(\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}(-t) \right), \\ \cos(|\xi|t) &= \frac{|\xi|}{2} \left(\hat{\mathcal{U}}(t) + \hat{\mathcal{U}}(-t) \right), \end{aligned}$$

it follows

$$\begin{aligned} \mathcal{R}(t) * u_1 &= \frac{1}{2} \left(\mathcal{U}(t) - \mathcal{U}(-t) \right) * u_1, \\ \partial_t \mathcal{R}(t) * u_0 &= \frac{1}{2} (-\Delta)^{1/2} \left(\mathcal{U}(t) + \mathcal{U}(-t) \right) * u_0, \end{aligned} \tag{56}$$

and so in fact estimates on $\mathcal{U}(t)*(\cdot)$ give estimates on the homogeneous solution $\underline{u}$. Conversely, since $\mathcal{U}(t)*g$ is a solution to the homogeneous equation with initial data $u_0 = (-\Delta)^{-1/2}g$, $u_1 = ig$, it follows that all estimates so far hold also for $\mathcal{U}(t)*(\cdot)$.

As an application of the above, the final estimate is an elegant L^q-bound which is global in space and time.

Theorem 1.11 (Scattering estimate)**.** *Solutions to the homogeneous wave equation* (4) *satisfy*

$$\|\underline{u}\|_{L^q(\mathbb{R}^{n+1})} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} + \|u_1\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)}), \tag{57}$$

where $q = 2\frac{n-1}{n+1}$.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing given by integration, and let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$. Then

$$\begin{aligned} |\langle \mathcal{U}(\cdot) * g, \eta \rangle| &= \left| \iint \frac{e^{i|\xi|t}}{|\xi|} \hat{g}(\xi) \overline{\hat{\eta}(\xi,t)} d\xi dt \right| \\ &= \left| \int |\xi|^{-\frac{1}{2}} \hat{g}(\xi) \int \overline{\frac{e^{-i|\xi|t}}{|\xi|^{\frac{1}{2}}} \hat{\eta}(\xi,t) dt} d\xi \right| \\ &\leq \|g\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)} \left\| \int \frac{e^{-i|\cdot|t}}{|\cdot|^{\frac{1}{2}}} \hat{\eta}(\cdot,t) dt \right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{58}$$

The second term may be estimated as follows.

$$\begin{aligned} \left\| \int \frac{e^{-i|\cdot|t}}{|\cdot|^{\frac{1}{2}}} \hat{\eta}(\cdot,t) dt \right\|^2_{L^2(\mathbb{R}^n)} &= \int (\int \frac{e^{-i|\xi|s}}{|\xi|^{\frac{1}{2}}} \hat{\eta}(\xi,s) ds) (\int \overline{\frac{e^{-i|\xi|t}}{|\xi|^{\frac{1}{2}}} \hat{\eta}(\xi,t) dt}) d\xi \\ &= \iint \left\langle \frac{e^{i|\cdot|(t-s)}}{|\cdot|} \hat{\eta}(s), \hat{\eta}(t) \right\rangle dsdt \\ &= \iint \langle \mathcal{U}(t-s) * \eta(s), \eta(t) \rangle \, dsdt \\ &\leq \int \|\eta(t)\|_{L^p} \int \|\mathcal{U}(t-s) * \eta(s)\|_{L^q} dsdt. \end{aligned} \tag{59}$$

Next, we note that $\dot{B}^0_{q,r} \hookrightarrow L^q$ whenever $q > r$ and $L^p \hookrightarrow \dot{B}^0_{p,r}$ whenever $p < r$. Thus from Lemma 1.10, whenever $\frac{n-1}{2} - \frac{n+1}{q} = 0$, i.e. $q = 2\frac{n-1}{n+1} < 2$, we have

$$\|\mathcal{U}(t-s) * \eta(s)\|_{L^q} \leq \|\mathcal{U}(t-s) * \eta(s)\|_{\dot{B}^0_{q,2}} \leq \frac{C}{(t-s)^{\frac{n-1}{n+1}}} \|\eta(s)\|_{\dot{B}^0_{p,2}} \leq \frac{C}{(t-s)^{\frac{n-1}{n+1}}} \|\eta(s)\|_{L^p},$$

so defining

$$f(s) = \|\eta(s)\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad I_\alpha(f)(t) = \int_{-\infty}^{\infty} \frac{f(s)}{|s-t|^{\frac{1}{\alpha}}} ds,$$

for $\frac{1}{\alpha} = \frac{n-1}{n+1} < 1$, this gives, by the Hardy-Littlewood-Sobolev inequality, see page 23 of [13],

$$\begin{aligned} \left\| \int \frac{e^{-i|\cdot|t}}{|\cdot|^{\frac{1}{2}}} \hat{\eta}(t) dt \right\|^2_{L^2(\mathbb{R}^n)} &\leq \int f(t) \int \frac{f(s)}{(t-s)^{\frac{n-1}{n+1}}} dsdt \\ &= \int f(t) I_\alpha(f)(t) dt \\ &\leq \|f\|_{L^p(\mathbb{R})} \|I_\alpha(f)\|_{L^q(\mathbb{R})} \\ &\leq \|f\|_{L^p(\mathbb{R})} \|f\|_{L^{q'}}, \end{aligned} \tag{60}$$

where $\frac{1}{\alpha} = 1 - (\frac{1}{q'} - \frac{1}{q})$. Amazingly, for our case $\frac{1}{\alpha} = \frac{n-1}{n+1} = \frac{2}{q}$, this gives $q' = p$ and so from (58) we arrive at

$$|\langle \mathcal{U}(\cdot) * g, \eta \rangle| \leq C \|u_1\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)} \|\eta\|_{L^p(\mathbb{R}^{n+1})}.$$

Hence it follows

$$\|\mathcal{U}(\cdot) * g\|_{L^q(\mathbb{R}^{n+1})} \leq C \|g\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)},$$

which implies, moreover,

$$\|(-\Delta)^{1/2}(\mathcal{U}(\cdot) * g)\|_{L^q(\mathbb{R}^{n+1})} = \|\mathcal{U}(\cdot) * (-\Delta)^{1/2} g\|_{L^q(\mathbb{R}^{n+1})} \leq C \|g\|_{\dot{H}^{\frac{1}{2}}},$$

which by (56) gives the desired result. □

Theorem (1.11) was proved first by Strichartz in [14]. More general scattering estimates are given by Keel and Tao in [7], by interpolation with the energy estimates. In particular, one result of Keel and Tao is as follows.

Theorem 1.12. *For all $q, r \geq 2$, where $(q, r) \neq (2, \infty)$ when $n = 3$ and where*

$$\left(\frac{2}{n-1}\right) \frac{1}{q} + \frac{1}{r} = \frac{1}{2},$$

solutions to the homogeneous wave equation (4) *satisfy*

$$\|\underline{u}\|_{L^q([0,T);L^r(\mathbb{R}^n))} \leq C \left(\|u_1\|_{H^{-1}(\mathbb{R}^n)} + \|u_0\|_{L^2(\mathbb{R}^n)}\right) \tag{61}$$

2 Nonlinear wave equations

For this section, the book by Evans [2] was helpful in writing the fixed point arguments. Moreover, the proof of global existence in the sub-critical energy case was taken from Evans. Other sources are referenced as they appear.

2.1 Local existence

Turning to the non-linear Cauchy problem, we will consider the *semi-linear* case, where the nonlinearity depends on u but not on its derivatives.

$$\begin{cases} \Box u = f = f(t, u), \\ u(0) = u_0; \quad \partial_t u(0) = u_1. \end{cases} \tag{62}$$

Definition 2.1.

- If $u\colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ solves (62), in the classical or weak sense, we say that u is a *global solution*. We talk of the existence of a global solution as existence for all-time.

- If $u\colon \mathbb{R}^n \times [0,T) \to \mathbb{R}$ solves (62) for some $T>0$, in the classical or weak sense, we say that u is a *local solution.* We talk of the existence of a local solution as short-time existence (up to time T).

- For the case that a Cauchy problem is known to admit unique local solutions, but no global solution, we refer to as *blow-up* of solution. In this case, the time

$$T^* = \sup\{T>0\colon \text{ a local solution exists up to time } T\} \tag{63}$$

is referred to as the blow-up time.

The existence of local solutions may be established using the energy estimates. Let the notation

$$\|v(t)\|^2_{E^s(\mathbb{R}^n)} = \|v(t)\|^2_{H^s(\mathbb{R}^n)} + \|v_t(t)\|^2_{H^{s-1}(\mathbb{R}^n)} \tag{64}$$

be introduced for an energy norm.

Theorem 2.2 (Short time existence)**.** *If the nonlinearity $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies $f\in\mathcal{C}^k$, where $k > n/2$, and $f(0,0)=0$, then given initial data $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}$ for $s > n/2$, then the Cauchy problem* (62) *admits a unique local solution, in the weak sense, with $u \in L^\infty([0,T];E^s(\mathbb{R}^n))$.*

Proof. For a given $v \in L^\infty([0,T];L^2(\mathbb{R}^n))$, let $u = \Psi[v]$ denote the unique, weak solution to the linear system

$$\begin{cases} \Box u = f(t,v), \\ u(0) = u_0; \ \ \partial_t u(0) = u_1, \end{cases}$$

so as in (7) we have

$$\Psi[v] = \underline{u} + \mathcal{L}(f(t,v)).$$

Let $E_0 = \|u_0\|_{H^s(\mathbb{R}^n)} + \|u_1\|_{H^{s-1}(\mathbb{R}^n)}$ be an initial energy and define

$$B = B(T) = \{v \in L^\infty([0,T];L^2(\mathbb{R}^n))\colon \sup_{0\le t\le T} \|v(t)-u_0\|_{E^s(\mathbb{R}^n)} \le 2E_0+1\}, \tag{65}$$

which is a bounded subset of the Banch space $L^\infty([0,T];L^2(\mathbb{R}^n))$, consisting of points bounded by the stronger norm of $L^\infty([0,T];E^s(\mathbb{R}^n))$.

Since f is $\mathcal{C}^k$, it follows that f is locally Lipschitz, and as $H^s \hookrightarrow \mathcal{C}^0$ for $s > n/2$ by Sobolev embedding, it follows that there exists a K_0 such that for any $v,\hat{v}\in B$,

$$|f(t,v) - f(t,\hat{v})| \le K_0|v-\hat{v}|. \tag{66}$$

Moreover, it follows from the Sobolev embedding, since $f\in\mathcal{C}^k$ for $k > n/2$ and since $f(0,0)=0$, that there is a continuous and non-decreasing function $\mathcal{Q}\colon [0,\infty)\to[0,\infty)$ such that

$$\|f(t,v)\|_{L^\infty([0,T];E^s(\mathbb{R}^n))} \le \mathcal{Q}(\|v\|_{L^\infty([0,T];E^s(\mathbb{R}^n))}), \tag{67}$$

see [2], theorem 2, pg.666.

We claim that for $T > 0$ sufficiently small $\Psi\big|_B : B \to B$ is well defined and a contraction mapping. Indeed, by (35), (36) and (67), we see that for $v \in B$

$$\begin{aligned}\|\Psi[v] - u_0\|_{L^\infty([0,T];E^s(\mathbb{R}^n))} &\leq \|\underline{u} - u_0\|_{L^\infty([0,T];E^s(\mathbb{R}^n))} + \|\mathcal{L}(f(t,v))\|_{L^\infty([0,T];E^s(\mathbb{R}^n))} \\ &\leq \|\underline{u}\|_{L^\infty([0,T];E^s(\mathbb{R}^n))} + \|u_0\|_{H^s(\mathbb{R}^n)} + T\|f(t,v)\|_{L^\infty([0,T];E^s(\mathbb{R}^n))} \\ &\leq 2E_0 + T\mathcal{Q}(\|v\|_{L^\infty([0,T];E^s(\mathbb{R}^n))}) \leq 2E_0 + 1,\end{aligned} \tag{68}$$

provided it is chosen $T \leq \mathcal{Q}(2E_0 + 1)$, and so $\Psi\big|_B : B \to B$ is well defined.

Furthermore, for $v, \hat{v} \in B$, by (36) and (66)

$$\begin{aligned}\|\Psi[v] - \Psi[\hat{v}]\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} &= \|\mathcal{L}(f(t,v) - f(t,\hat{v}))\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} \\ &\leq T\|f(t,v) - f(t,\hat{v})\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} \\ &\leq TK_0\|v - \hat{v}\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} \leq \frac{1}{2}\|v - \hat{v}\|_{L^\infty([0,T];L^2(\mathbb{R}^n))},\end{aligned} \tag{69}$$

provided it is chosen $T \leq \frac{1}{2K_0}$, and so $\Psi\big|_B$ is a contraction with respect to the weaker norm on $L^\infty([0,T];L^2(\mathbb{R}^n))$.

Now, choosing $T = \min\{\mathcal{Q}(2E_0+1), \frac{1}{2K_0}\}$ and defining iteratively $u_k = \Psi[u_{k-1}]$, by the contraction mapping principle we have that u_k converges to a unique fixed point u of the map Ψ, which is the unique, weak solution.

Since this convergence is with respect to the weaker norm of $L^\infty([0,T];L^2(\mathbb{R}^n))$, a priori we have no extra regularity on the solution u. However, by the Banach-Alaoglu theorem, since $\{u_k\}$ is bounded in $L^\infty([0,T];E^s(\mathbb{R}^n))$, and the weak-$\star$ topology is metrizable here (as the dual of a separable space), a weak-$\star$ convergent subsequence may be extracted. Then by weak-$\star$ lower-semicontinuity of the norm, we see in fact that $u \in L^\infty([0,T];E^s(\mathbb{R}^n))$ as desired. □

Examining the above proof, we arrive at the following necessary condition for solution blow-up

Corollary 2.3 (H^s-criterion for blow-up)**.** *If the Cauchy problem* (62) *exhibits blow up at a finite time* $T^* > 0$*, then*

$$\limsup_{t \to T^*} \|u(t)\|_{H^s(\mathbb{R}^n)} = \infty. \tag{70}$$

Proof. Suppose, on the contrary, that $\|u(t)\|_{H^s(\mathbb{R}^n)}$ stays bounded as $t \to T^*$. Then we may find ε so that

$$\sup_{T^*-\varepsilon \leq t \leq T^*} \|u(t)\|_{H^s(\mathbb{R}^n)} \leq C_1 < \infty.$$

Then, reposing the Cauchy problem (62) with $u_0 = u(T^* - \varepsilon), u_1 = u_t(T^* - \varepsilon)$, we have a solution, to which we reallocate the label u, which blows up at time ε and satisfies

$$\sup_{0 \leq t \leq \varepsilon} \|u(t)\|_{H^s(\mathbb{R}^n)} \leq C_1 < \infty.$$

Now, a K_0 may be chosen so that

$$|f(t,z)-f(t,\hat{z})| \le K_0|z-\hat{z}| \text{ whenever } (t,z) \in [0,2\varepsilon]\times[-(2C_1+E_0+1),(2C_1+E_0+1)].$$

Then defining $T = \min\{\mathcal{Q}(E_0+C_1+1), \frac{1}{2K_0}, \frac{\varepsilon}{2}\}$ and

$$B_k = \{v \in L^\infty([kT,(k+1)T];L^2(\mathbb{R}^n))\colon \sup_{kT\le t\le (k+1)T} \|v(t)-u(kT)\|_{H^s(\mathbb{R}^n)} \le E_0+C_1+1\},$$

and letting $w = \Psi_k[v]$ be the unique solution to

$$\begin{cases} \Box w = f(t,v), \\ w(0) = u(kT); \quad \partial_t w(0) = \partial_t u(kT), \end{cases}$$

we have

$$\begin{aligned} \|\Psi[v]-u(kT)\|_{L^\infty([0,T];H^s(\mathbb{R}^n))} &\le \|\underline{u}-u(kT)\|_{L^\infty([0,T];H^s(\mathbb{R}^n))} + \|\mathcal{L}(f(t,v))\|_{L^\infty([0,T];H^s(\mathbb{R}^n))} \\ &\le E_0 + C_1 + T\|f(t,v)\|_{L^\infty([0,T];H^s(\mathbb{R}^n))} \le E_0+C_1+1, \end{aligned}$$

and

$$\|\Psi[v]-\Psi[\hat{v}]\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} \le T\|f(t,v)-f(t,\hat{v})\|_{L^\infty([0,T];L^2(\mathbb{R}^n))},$$

thus $\Psi\big|_{B_k} \colon B_k \to B_k$ is a well defined contraction map for any k, allowing us to continue the solution past time ε, and contracting the assumption of blow-up. $\square$

For the case of $n = 3$, we are able to improve on this criterion.

Theorem 2.4 (L^∞-criterion for blow-up)**.** *If $n = 3$ and the Cauchy problem* (62) *exhibits blow-up at a finite time $T^* > 0$, then*

$$\limsup_{t\to T^*} \|u(t)\|_{L^\infty(\mathbb{R}^3)} = \infty.$$

Proof. The proof is to note that we can establish existence via a contraction in the L^∞-topology. Indeed, letting

$$B = \{v \in L^\infty([0,T]\times\mathbb{R}^3)\colon \sup_{0\le t\le T} \|v(t)-u_0\|_{L^\infty(\mathbb{R}^3)} = \infty\},$$

then for T sufficiently small, by the representation formula (16),

$$\begin{aligned} \|\Psi[v]-u_0\|_{L^\infty([0,T]\times\mathbb{R}^3)} &\le \|\underline{u}(t)-u_0\|_{L^\infty([0,T]\times\mathbb{R}^3)} + \left\| \left(\int_{B_t(x)} |y|^{-1} f(t-|y|, v(y,t-|y|))dy \right) \right\|_{L^\infty([0,T]\times\mathbb{R}^3)} \\ &\le 2\|u_0\|_{L^\infty(\mathbb{R}^3)} + \left(\int_{B_T(x)} |y|^{-1} dy \right) \|f(t,v)\|_{L^\infty([0,T]\times\mathbb{R}^3)} \\ &\le 2\|u_0\|_{L^\infty(\mathbb{R}^3)} + \pi^2 T^2 C_0 \le 2\|u_0\|_{L^\infty(\mathbb{R}^3)} + 1, \end{aligned}$$

and moreover

$$
\begin{aligned}
\|\Psi[v]-\Psi[\hat{v}]\|_{L^\infty([0,T]\times\mathbb{R}^3)} &\leq \left(\int_{B_T(x)}|y|^{-1}dy\right)\|f(t,v)-f(t,\hat{v})\|_{L^\infty([0,T]\times\mathbb{R}^3)}\\
&\leq \pi^2T^2K_0\|v-\hat{v}\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq \frac{1}{2}\|v-\hat{v}\|_{L^\infty([0,T]\times\mathbb{R}^3)},
\end{aligned}
$$

so $\Psi\colon B\to B$ in an L^∞-contraction. Since a uniform L^∞ bound lets us choose C_0, K_0 and hence T, uniformly as in the proof of Corollary 2.3, the criterion follows similarly. ☐

2.2 Energy criticality

We shall consider from now only equations which govern autonomous systems, where the nonlinearity is independent of time, $f(t,u)=f(u)$. An important feature of these semi-linear problems is a conserved energy functional, analagous to the linear case. For $F(u) := \int_u f(v)dv$, we define an energy density by

$$
e(u(t)) = \frac{1}{2}\left(|u_t(t)|^2+|\nabla u(t)|^2\right) - F(u). \tag{71}
$$

Multiplying the equation (62) by u_t gives

$$
\begin{aligned}
0 &= (\Box u - f(u))u_t\\
&= \frac{d}{dt}\left(\frac{1}{2}\left(|u_t(t)|^2+|\nabla u(t)|^2\right)-F(u)\right) - \operatorname{div}(u_t\nabla u),
\end{aligned} \tag{72}
$$

and so writing

$$
\begin{aligned}
E(u(t)) &= \int_{\mathbb{R}^3} e(u(t))dx\\
E(u;\Omega(t)) &= \int_{\Omega(t)} e(u(t))dx
\end{aligned} \tag{73}
$$

for the energy, we have firstly that $\frac{d}{dt}E(u(t))=0$ for initial data $(u_0,u_1)\in\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Secondly, integrating (72) over the section of light-cone $C_s^t(x_0,t_0)$ gives

$$
\begin{aligned}
0 &= \int_{C_s^t(x_0,t_0)}\left(\frac{d}{dt}\left(e(u(t))\right)-\operatorname{div}(u_t\nabla u)\right)dxdt\\
&= \frac{d}{dt}\left(\int_s^t\int_{D_{(x_0,t_0)}(\tau)} e(u(\tau))dxdt\right) - \frac{1}{\sqrt{2}}\int_{\mathcal{M}_s^t(x_0,t_0)} e(u(\tau))d\omega(x,\tau) - \int_s^t\int_{D_{(x_0,t_0)}(\tau)} u_t\nabla u\cdot\frac{x}{|x|}d\sigma_\tau(x)d\tau\\
&= E(u;D_{(x_0,t_0)}(t)) - E(u;D_{(x_0,t_0)}(s)) + \frac{1}{\sqrt{2}}\int_{\mathcal{M}_s^t(x_0,t_0)}\left(e(u(t))-u_t\left(\nabla u\cdot\frac{x}{|x|}\right)\right)d\omega.
\end{aligned} \tag{74}
$$

Now, if we define $v(y)=u(x_0+y,t_0-|y|)$, then the flux integrand becomes

$$
e(u(t))-u_t\left(\nabla u\cdot\frac{x}{|x|}\right) = \frac{1}{2}\left|\nabla u - u_t\frac{x}{|x|}\right|^2 - F(u) = \frac{1}{2}|\nabla v|^2 - F(v),
$$

and so we may state the energy-flux identity as follows.

Lemma 2.5 (Energy-flux identity)**.** *For any $s < t$, it holds that*

$$E(u; D_{(x_0,t_0)}(t)) + \int_{B_{t_0-s}\setminus B_{t_0-t}} \left(\frac{1}{2}|\nabla v|^2 - F(v)\right) dy = E(u; D_{(x_0,t_0)}(s)). \tag{75}$$

We will now introduce the notion of *energy criticality* via a toy problem; an elliptic PDE

$$-\Delta u = f(u), \tag{76}$$

where the nonlinearity $f \in \mathcal{C}^k(\mathbb{R} \to \mathbb{R})$ is some function obeying the growth condition

$$\begin{aligned} |f(u)| &\leq C|u|^p, \\ |f^{(j)}(u)| &\leq C|u|^{q(j)} \text{ for all } j \in \{1, \dots, k\}, \end{aligned} \tag{77}$$

for some values $q(j)$. That is, f and its derivatives have at most polynomial growth. Let us consider this PDE with domain the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ (the advantage of a closed manifold being that we may avoid having to introduce a function to cut-off at the boundary) and let us suppose that we are *given* an a-priori H^1-bound on the solution.

Multiplying through (76) by u gives

$$\int_{T^n} -\Delta u \cdot u dx = \int_{T^n} f(u) \cdot u dx,$$

and from an integration by parts it follows

$$\int_{T^n} |\nabla u|^2 dx \leq C \int_{T^n} |u|^{p+1} dx,$$

which we note, when $p < 2^* - 1$ is stronger than the Sobolev embedding. Moreover, multiplying by u^α, where $\alpha > 1$, and integrating by parts gives

$$\frac{\alpha}{(\alpha+1)^2} \int_{T^n} |\nabla \left(u^{\frac{\alpha+1}{2}}\right)|^2 dx \leq C \int_{T^n} |u|^{p+\alpha} dx. \tag{78}$$

Now, the Sobolev embedding $W^{1,2} \hookrightarrow L^{2^*}$ and the Poincaré inequality give $\|u\|_{L^{2^*}(T^n)} \leq C(T^n)\|\nabla u\|_{L^2(T^n)}$, which applied to (78) imply

$$\left(\int_{T^n} |u|^{(\alpha+1)\frac{n}{n-2}} dx\right)^{\frac{n-2}{n}} \leq C\alpha \left(\int_{T^n} |u|^{p+\alpha} dx\right). \tag{79}$$

Thus choosing

$$p < 2^* - 1, \tag{80}$$

and (79) reads $\|u\|_{L^{\lambda q}(T^n)} \leq C(q)\|u\|_{L^q(T^n)}$ for

$$\lambda = \left(\frac{n}{n-2}\right)\left(\frac{1+\alpha}{p+\alpha}\right) > \left(\frac{n}{n-2}\right)\left(\frac{2}{p+1}\right) = \frac{2^*}{p+1} \geq 1 + \varepsilon(p) > 1,$$

and all $q \geq 2^*$.

Thus the a-priori bound $u \in H^1$ implies $u \in L^q$ for all q, and hence $f^{(j)}(u) \in L^q$ for all q, all $j \in \{0, 1, \dots, k\}$.

Now, since $-\Delta v = f$ for a function $f \in H^s$ implies $v \in H^{s+2}$, so it follows that $f(u) \in L^2$ implies $u \in H^2$, so $u \in W^{1,2^*}$ by Sobolev embedding. Differentiating (76) then gives

$$\Delta(\nabla u) = f'(u)\nabla u,$$

and by Hölder

$$\int_{T^n} (f'(u)\nabla u)^2 dx \leq C \left(\int_{T^n} f'(u)^{\frac{2^*}{2^*-2}} dx \right)^{\frac{2^*-2}{2^*}} \left(\int_{T^n} |\nabla u|^{2^*} dx \right)^{\frac{1}{2^*}} \leq C,$$

so $\nabla u \in W^{1,2}$ implies $u \in W^{3,2}$, so $u \in W^{2,2^*}$ by Sobolev embedding, and differentiating again and applying Hölder and so on will give $u \in H^s$ for all $s \leq k+2$. We thus see that an a priori H^1-bound induces bounds on higher norms.

Heuristically then, we may hope for the Cauchy problem (62), where the nonlinearity satisfies the growth condition (77), that control of the H^1-norm uniformly in time will imply control of the higher H^s-norms, in particular ensuring global existence for solutions by Corollary 2.3. For certain Cauchy problems, specifically those for which $F(u) := \int^u f(v)dv \leq 0$ such an a-priori H^1-bound is given by energy conservation.

In particular, one such toy equation we shall consider is the *defocussing* wave equation.

$$\begin{cases} \Box u = -|u|^{p-1}u, \\ u(0) = u_0; \;\; \partial_t u(0) = u_1, \end{cases} \tag{81}$$

for which the energy functional is given by

$$E(u(t)) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{p+1}|u|^{p+1} \right) dx. \tag{82}$$

We will discuss some results for this equation in the case of $n = 3$. We will be concerned with the question of global existence given smooth, compactly supported initial data.

Theorem 2.6 (Global existence in the subcritical case, and for small energy in the critical case.)**.** *Given initial data $u_0, u_1 \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ solutions to the defocussing wave equation* (81) *in $n = 3$ dimensions exist for all time when $1 \leq p < 5$. When p=5, solutions will exist for all time so long as the initial energy satisfies*

$$E_0 = \int_{\mathbb{R}^3} \left(\frac{1}{2}|u_1|^2 + \frac{1}{2}|\nabla u_0|^2 + \frac{1}{6}|u_0|^6 \right) dx \leq \varepsilon_0,$$

for some $\varepsilon_0 > 0$.

Proof. We note firstly that, since the initial data is compactly supported, we may assume without loss of generailty, having relabelled the origin as necessary, that a solution u up to time T takes its supremum along the line $x = 0$. From the representation formula (16), it follows by Hölder's inequality that solutions satisfy

$$\begin{aligned} |u(0,t)| &\leq |\underline{u}(0,t)| + \int_{B_t} \frac{|v(y)|^p}{|y|} dy \\ &\leq |\underline{u}(0,t)| + \left(\int_{B_t} \frac{|v|^2}{|y|^2} dy \right)^{\frac{1}{2}} \left(\int_{B_t(x)} |v|^{2(p-1)} dy \right)^{\frac{1}{2}}, \end{aligned} \tag{83}$$

where $v(y) = u(y, t - |y|)$.

Starting with the critical case $p = 5$, Hardy's inequality, Lemma A.2, gives

$$\begin{aligned} |u(0,t)| &\leq |\underline{u}(0,t)| + C\left(\int_{B_t} |\nabla v|^2 dx + \left(\int_{B_t} |v|^6 dx\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \left(\int_{B_t} |v|^8 dy\right)^{\frac{1}{2}} \\ &\leq |\underline{u}(0,t)| + CE_0^{\frac{1}{2}} \left(\int_{B_t} |v|^6 dy\right)^{\frac{1}{2}} \|u\|_{L^\infty(\mathbb{R}^3\times[0.T])} \\ &\leq |\underline{u}(0,t)| + CE_0 \|u\|_{L^\infty(\mathbb{R}^3\times[0,t])}, \end{aligned} \tag{84}$$

and thus there is $\varepsilon_0 > 0$ such that $\|u(t)\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded provided $E_0 \leq \varepsilon_0$ and we deduce global existence by the blow-up criterion, Theorem 2.4.

For the critical case $1 \leq p < 5$, we note from the proof of Hardy's inequality (139) that we have

$$\int_{B_t} \frac{|v|^2}{|y|^2} dy \leq C\left(\int_{B_t} |\nabla v|^2 dy + \frac{1}{t^2}\int_{B_t} |v|^2 dy\right).$$

Now, Poincaré's inequality says

$$\int_{B_t} |v - \bar{v}|^2 dy \leq Ct^2 \int_{B_t} |\nabla v|^2 dy,$$

and so

$$\frac{1}{t^2}\int_{B_t} |v|^2 dy \leq C\int_{B_t} |\nabla v|^2 dy + t|\bar{v}|,$$

where the average $\bar{v}$ may be estimated as

$$\begin{aligned} |\bar{v}| &= \frac{C}{t^3}\int_{B_t} v(y)dy \\ &= \frac{C}{t^3}\int_{M_0^t} u(y)d\omega(y) \\ &= \frac{C}{t^3}\left(\int_{C_0^t} u_t(y)dy + \int_{B_t} u_0(y)dy\right) \\ &\leq \frac{C}{t}\left(\int_{C_0^t} u_t(y)^2\right)^{\frac{1}{2}} + C\|u_0\|_{L^\infty(\mathbb{R}^3)} \\ &\leq \frac{C}{t^{1/2}} E_0^{\frac{1}{2}} + C\|u_0\|_{L^\infty(\mathbb{R}^3)}, \end{aligned}$$

and so

$$\int_{B_t} \frac{|v|^2}{|y|^2} dy \leq C\left(\int_{B_t} |\nabla v|^2 dy + t^{\frac{1}{2}} E_0^{\frac{1}{2}} + t\|u_0\|_{L^\infty}\right). \tag{85}$$

Thus applying the energy-flux identity, Lemma 75, the estimate (83) gives

$$|u(0,t)| \leq |\underline{u}(0,t)| + C(E_0 + t^{\frac{1}{2}} E_0 + t\|u_0\|_{L^\infty})^{\frac{1}{2}} \left(\int_{B_t} |v|^{2(p-1)} dy\right)^{\frac{1}{2}}. \tag{86}$$

Firstly note, when $1 \leq p \leq 4$ we have by Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^6$, and by the energy-flux identity,

$$\begin{aligned} \int_{B_t} |v|^{2(p-1)} dy &\leq \left(\left(\int_{B_t} dy \right)^{\frac{4-p}{3}} \left(\int_{B_t} |v|^6 dy \right)^{\frac{2(p-1)}{6}} \right)^{2(p-1)} \\ &\leq C t^{2(p-1)(4-p)} \|v\|_{H^1}(B_t)^{4(p-1)^2} \\ &\leq C t^{2(p-1)(4-p)} E_0^{2(p-1)^2}, \end{aligned} \tag{87}$$

so $\|u\|_{L^\infty([0,T]\times\mathbb{R}^3)}$ is bounded.

For the case $4 < p < 5$ we then have

$$\begin{aligned} |u(0,t)| &\leq |\underline{u}(0,t)| + C(E_0 + t^{\frac{1}{2}} E_0 + t\|u_0\|_{L^\infty})^{\frac{1}{2}} \left(\int_{B_t} |v|^6 dy \right)^{\frac{1}{2}} \|u\|^q_{L^\infty([0,t]\times\mathbb{R}^3)} \\ &\leq |\underline{u}(0,t)| + C\|u\|^q_{L^\infty([0,t]\times\mathbb{R}^3)}), \end{aligned} \tag{88}$$

where $0 < q < 1$, so $\|u\|_{L^\infty([0,T]\times\mathbb{R}^3)}$ is bounded.

Thus we have global existence for the subcritical problem. □

2.3 Global solutions for the energy-critical, defocussing equation in 3D

In this section we will present the argument of [15] in which global existence of radially symmetric solutions is proved for the energy-critical, defocussing wave equation

$$\begin{cases} \Box u = -u^5, \\ u(0) = u_0; \quad \partial_t u(0) = u_1, \end{cases} \tag{89}$$

in $n = 3$ space dimensions.

Theorem 2.7 (Struwe)**.** *Given initial data $u_0 \in \mathcal{C}^3(\mathbb{R}^3), u_1 \in \mathcal{C}^2(\mathbb{R}^3)$ which is radially symmetric, i.e. $u_0(x) = u_0(|x|), u_1(x) = u_1(|x|)$, the energy critical, defocussing wave equation* (89) *admits a unique, global solution $u \in \mathcal{C}^2(\mathbb{R}^3 \times [0,\infty))$ which is radially symmetric, i.e. $u(x,t) = u(|x|,t)$.*

The existence of a unique, local solution with the desired regularity is given by the earlier fixed point arguments. To see that such a solution is radially symmetric, let $A \in O(3)$ be an orthogonal matrix. Since A commutes with Δ and hence with $\Box$, it follows that for any u_0, u_1, defining $\hat{u}_0(x) = u_0(Ax)$, $\hat{u}_1(x) = u_1(Ax)$ and $\hat{u}(x,t) = u(Ax,t)$ we have $(\Box\hat{u})(x,t) = (\Box u)(Ax,t)$ and so $\hat{u}$ solves the Cauchy problem with initial conditions $\hat{u}_0, \hat{u}_1$. Thus by the uniqueness of solutions, radially symmetric initial data must give a radially symmetric solution.

Next observe the following corollary of the short-time existence for small energy from Theorem 2.6. Since the L^∞-norm must become unbounded approaching a blow up by Lemma 2.4, it follows by the energy-flux identity and Theorem 2.6 that Energy concentration is a necessary condition for solution blow-up.

Corollary 2.8 (Energy concentration criterion)**.** *If u is a solution to* (89) *in $n = 3$ space dimensions, which blows up at a time T^*, then*

$$E(u; D_{(x_0,T^*)}(T^* - \delta)) \geq \varepsilon_0 > 0, \tag{90}$$

for any $\delta > 0$.

Struwe's argument supposes the existence of a blow-up time T^*, and hence a point (x_0, T^*) at which

$$\lim_{(x,t)\to(x_0,T^*)} |u(x,t)| = \infty,$$

where energy must necessarily concentrate. Firstly it is noted that for radially symmetric solutions, this singularity must arise at $x_0 = 0$. Indeed, supposing $|x_0| > 0$, then by radial symmetry $|u(x,t)| \to \infty$ whenever $(x,t) \to (x_j, T^*)$ for any $|x_j| = |x_0|$. Since arbitrarily many distinct such points can be found, and at least one quanta of energy ε_0 concentrates at any such point, this implies an infinite amount of energy, a contradiction.

Without loss of generality the point (x_0, T^*) is then taken to be the origin $(0, 0)$, having reposed the Cauchy problem to give a solution $u\colon \mathbb{R}^3 \times [-T^*, 0) \to \mathbb{R}$. A contradiction is reached by exploiting the scaling invariance of the equation to analyse the singularity. The following a-priori estimate is key.

Lemma 2.9. *For any solution $u\colon \mathbb{R}^3 \times [-T, 0) \to \mathbb{R}$ of* (89)*, there holds*

$$\frac{1}{3}\int_{C^0_{-1}} |u|^6 dxdt + E(u; D(-1)) \leq \int_{D(-1)} u_t \left(x \cdot \nabla u + u\right) dx + \int_{B_1} \left(|y||\nabla v|^2 + |\nabla v||v|\right) dy, \tag{91}$$

where $v(y) = u(y, -|y|)$.

Proof. We refer for proof to the original paper [15]. The trick is to test the equation against

$$\left.\frac{d}{dR}\right|_{R=1} u_R(x,t) = tu_t + x \cdot \nabla u + \frac{1}{2}u, \tag{92}$$

which is the generator of a family of solutions $\{u_R(x,t) = R^{\frac{1}{2}} u(Rx, Rt)\}$. ☐

The blow-up analysis involves rescaling the solution about the singularity, defining

$$u_m(x,t) = R_m^{\frac{1}{2}} u(R_m x, R_m t), \tag{93}$$

where $R_m = 2^{-m} \to 0$. We note that the $u_m\colon \mathbb{R}^3 \times [T_m, 0) \to \mathbb{R}$, where $T_m = -2^{-m}T^*$, trace a family of solutions to the Cauchy problem (89).

Moreover, we note crucially that the energy is an invariant with respect to this rescaling, in the sense that

$$E(u_m; D(s)) = E(u; D(R_m s)). \tag{94}$$

Applying Lemma 2.9 to this sequence, we have

$$\begin{aligned}\frac{1}{3}\int_{C^0_{-1}}|u_m|^6dxdt+E(u_m;D(-1))\leq \overbrace{\int_{D(-1)}(u_t)\,(x\nabla u_m)\,dx}^{\textbf{(I)}} \\ +\underbrace{\int_{D(-1)}(u_m)_t(u_m)dx}_{\textbf{(II)}}+\underbrace{\int_{B_1}\left(|y||\nabla v_m|^2+|\nabla v_m||v_m|\right)dy}_{\textbf{(III)}},\end{aligned} \tag{95}$$

where $v_m(y)=u_m(y,-|y|)$.

Lemma 2.10. *Terms* **(II)** *and* **(III)** *in the above expression* (95) *are* $o(1)$*. Where* $o(1)$ *denotes a term tending to zero as* $m\to\infty$*.*

Proof. For term **(III)**, we note by energy invariance (94), the energy-flux identity and Corollary (2.8) that $E(u_m;D(-1))$ is decreasing and bounded below by ε_0. Thus $E(u_m;D(-1))\to\varepsilon_1\geq\varepsilon_0>0$, and moreover we see that $E(u_m;D(-\delta))\to\varepsilon_1\to\varepsilon_0$ for any $\delta>0$. So the energy-flux identity implies

$$\int_{B_1}\frac{1}{2}|\nabla v_m|^2+\frac{1}{6}|v_m|^6dy=o(1)$$

and so **(III)** is $o(1)$ by Hölder.

For the term **(II)**, we refer to Lemma 3.2 in [15]. ☐

Thus, rearranging (95) gives

$$\begin{aligned}\int_{C^0_{-1}}|u_m|^6dxdt+\int_{D(-1)}\left\{(1-|x|)\left(|(u_m)_t|^2+|\nabla(u_m)|^2\right)+|x|\left|\frac{x}{|x|}(u_m)_t-\nabla(u_m)\right|^2+|u_m|^6\right\}dx \\ \leq \textbf{(II)}+\textbf{(III)}=o(1)\end{aligned} \tag{96}$$

An immediate application of (96), if we define

$$D^\varepsilon(-1)=\{(x,-1)\in D(-1)\colon |x|\leq 1-\varepsilon\}, \tag{97}$$

is that for any $\varepsilon>0$,

$$\int_{D^\varepsilon(-1)}\left(\frac{1}{2}\left|(u_m)_t\right|^2+\frac{1}{2}\left|\nabla(u_m)\right|^2+\frac{1}{6}\left|(u_m)\right|^6\right)dx=o(1). \tag{98}$$

But it is necessary that energy concentrates at the singularity, so this suggests that energy concentrates "along a ring". The bulk of the proof is to show that this is an absurdity.

Lemma 2.11. *There exists a sequence* $\Lambda\subset\mathbb{N}$ *such that*

$$\liminf_{\substack{m\to\infty\\ m\in\Lambda}}\left\{\sup_{C^{-1}_{t_m}}|u_m|\right\}>0. \tag{99}$$

Proof. We refer to the original paper [15] for the proof of this lemma, which is quite technical. ☐

To conclude the proof from here, choose $(x_m, s_m) \in C_{T_m}^{-1}$ satisfying $|u_m(x_m, s_m)| = \sup_{C_{T_m}^{s_m}} |u_m|$, so by Lemma 2.11

$$\liminf_{\substack{m\to\infty\\ m\in\Lambda}} = c_0 > 0.$$

Since for any $s_0 \in [-T^*, 0)$, $\sup_{s\le s_0} |u(x,s)| = C(s_0) < \infty$, it follows that

$$\sup_{s\le s_0/R_m} |u_m(x,s)| = C(s_0) R_m^{\frac{1}{2}} \to 0,$$

and so in particular $R_m s_m \to 0$.

Note that for a given $s \in [T_m, s_m]$, by Hardy's inequality and the energy-flux identity

$$\begin{aligned}\frac{1}{\sqrt{2}} \int_{\mathcal{M}_s^{s_m}(x_m,s_m)} \left(\frac{u_m^4}{|\tau|}\right) d\omega(x,\tau) &= \int_{B_{|s|}} \left(\frac{v_m^4}{|y|}\right) dy \\ &\le \left(\int_{B_{|s|}} |\nabla v_m|^2 dy + \left(\int_{B_{|s|}} |v_m|^6 dy\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \left(\int_{B_{|s|}} |v_m|^6\right)^{\frac{1}{2}} \le \frac{1}{3},\end{aligned} \tag{100}$$

provided that $E(u; D_{(x_m,s_m)}(s)) \le \varepsilon_0$. Hence for sufficiently large m we have

$$\begin{aligned} 0 < \frac{1}{2}c_0 &\le \left(1 - \frac{1}{\sqrt{2}} \int_{\mathcal{M}_s^{s_m}(x_m,s_m)} \left(\frac{u_m^4}{|\tau|}\right) d\omega(z,\tau)\right) |u_m(x_m,s_m)| \\ &\le |\underline{u}_m(x_m,s_m)| + \frac{1}{\sqrt{2}} \int_{\mathcal{M}_{T_m}^{s_m}(x_m,s_m)} \left(\frac{|u_m|^5}{|\tau|}\right) d\omega(z,\tau) - \frac{1}{\sqrt{2}} \int_{\mathcal{M}_s^{s_m}(x_m,s_m)} \left(\frac{|u_m|^5}{|\tau|}\right) d\omega(z,\tau) \\ &= |R_m^{1/2} \underline{u}(x_m,s_m)| + \int_{B_{T_m-s_m}\setminus B_{s-s_m}} \left(\frac{|v_m|^5}{|y|}\right) dy \\ &\le o(1) + \left(\int_{B_{T_m-s_m}\setminus B_{s-s_m}} \frac{dy}{|y|^6}\right)^{\frac{1}{6}} \left(\int_{B_{T_m-s_m}\setminus B_{s-s_m}} v_m^6 dy\right)^{\frac{5}{6}} \\ &\le o(1) + C\frac{1}{|s-s_m|^{\frac{1}{2}}} E_0^{5/6}. \end{aligned} \tag{101}$$

But this becomes contradictory for large m and large $|s - s_m|$. Thus it follows that there exists c_1 such that

$$E(u_m; D_{(x_m,s_m)}(s)) > \varepsilon_0, \tag{102}$$

whenever $|s - s_m| \ge c_1$.

Next, Struwe observes the following consequence of the identity (98).

Lemma 2.12. *For any $c > 0$, and any family $\{x_m^k\}_{1\le k\le K} \in \mathbb{R}^3$ with $|x_m^k| = |x_m| \ge 0$, $|x_m^j - x_m^k| \ge c^{-1}|x_m|$, for m sufficiently large there exists $\sigma_m \in [t_m, s_m - c_6]$ such that*

$$E(u_m; \cup_{j\ne k} \left(D_{(x_m^j,s_m)}(\sigma_m) \cap D_{(x_m^k,s_m)}(\sigma_m)\right)) = o(1). \tag{103}$$

Proof. For a given K, since $|s_m| \ge 1$, and points x_m^k are distributed uniformly around the ring $|x| = |x_m|$, it follows that we may find $\varepsilon > 0$ independent of m so that

$$D_{(x_m^j,s_m)}(s) \cap D_{(x_m^k,s_m)}(s) \subset D^\varepsilon(s), \tag{104}$$

for $j \neq k$. But, by (98) we see

$$E(u_m; D^\varepsilon(s)) \leq E(u_{k(m)}; D^\varepsilon(-1)) = o(1), \tag{105}$$

for a $k(m)$ sufficiently large, which proves the claim. $\square$

And so, for any $K \in \mathbb{N}$, selecting K points x_m^k, such that $|x_m^j| = |x_m|$ and $|x_m^j - x_m^k| \geq c^{-1}|x_m|$ for all $1 \leq j \neq k \leq K$. By radial symmetry, $E(u_m; D_{(x_m^j, s_m)}(\sigma_m)) \geq \varepsilon_0$ for all j and so

$$\begin{aligned} K\varepsilon_0 &\leq \sum_{j=1}^K E(u_m; D_{(x_m^j,s_m)}(\sigma_m)) \\ &\leq E(u_m; \cup_{j=1}^K D_{(x_m^j,s_m)}(\sigma_m)) + \sum_{j\neq k} E(u_m; D_{(x_m^j,s_m)}(\sigma_m) \cap D_{(x_m^k,s_m)}(\sigma_m)) \\ &\leq E(u_m; D(\sigma_m)) + o(1) \leq E_0 + o(1), \end{aligned} \tag{106}$$

giving a contradiction for sufficiently large K. This concludes Struwe's proof of Theorem 2.7.

2.4 An example of blow-up in 3D

We have seen that the exponent $p = 2^* - 1$ is critical in the defocussing case, where the energy functional defines a norm. These heuristics are not sufficient for other wave equatons, however. F. John considered the Cauchy problem (62) in $n = 3$ space dimensions with a nonlinear term

$$f(u) \geq \alpha |u|^p, \tag{107}$$

for some $\alpha > 0$, and showed that for the case $1 < p < 1 + \sqrt{2}$, solutions with smooth, compactly supported initial data exhibit finite time blow up.

In this section, we review the first half of John's paper [5] which gives the proof of the following lemma.

Lemma 2.13. *Let u be a global solution to*

$$\begin{cases} \Box u \geq \alpha |u|^p, \\ u(0) = u_0\,;\ \ \partial_t u(0) = u_1, \end{cases} \tag{108}$$

where $\alpha > 0$ and $1 < p < \sqrt{2}$ and suppose that

$$\underline{u}(x,t) \geq 0 \quad \text{for } (x,t) \in \Gamma(x_0, t_0), \tag{109}$$

where $\underline{u}$ denotes the solution to the corresponding linear problem. Then u has compact support with

$$\operatorname{supp}(u) \subset C_0^{t_0}(x_0, t_0). \tag{110}$$

If we consider the Cauchy problem with initial data supported in a ball of radius R, it follows then from the Huygen's principle, Theorem 1.1, that $\underline{u}(x,t) = 0$ for $(x,t) \in \Gamma(0,R)$ for any global solution u, and so the Lemma applies here.

John remarks on the following consequence of Lemma 2.13. If a global solution u to (108) exists for compactly supported u_0, u_1, then necessarily it follows that

$$u_0(x) \geq 0 \text{ for all } x \in \mathbb{R}^3 \,; \tag{111}$$

$$\int_{\mathbb{R}^3} u_1(x)dx \leq 0. \tag{112}$$

Indeed, letting v be any solution of $\Box v = 0$ with initial data v_0, v_1 such that $v(x,t) \geq 0$, then we have

$$\begin{aligned} 0 \leq \int_{t>0} \alpha v|u|^p dxdt &\leq \int_{t>0} (v\Box u - u\Box v)dxdt \\ = -\int_{\mathbb{R}^3} (v_0 u_1 - v_1 u_0)dx, \end{aligned} \tag{113}$$

which gives (111) by choosing $v_0 = 1$, $v_1 = 0$ and respectfully gives (112) choosing $v_0 = 0$ and letting $v_1 \geq 0$ be arbitrary, having noted by the representation formula (16) that non-negative initial velocity leads to a non-negative solution.

In fact, by conservation of energy we can observe more. For the case where $f(0) \neq 0$, if a global solution u to (108) exists for compactly supported u_0, u_1, then necessarily it follows by conservation of energy that the initial energy is zero

$$E(u(0)) = \int_{\mathbb{R}^3} \left(\frac{1}{2} \left(|u_1|^2 + |\nabla u_0|^2 \right) - F(u_0) \right) dx = 0.$$

Remark 2.14. Note that for $t_0 > 0$, the function $u = a(t+t_0)^{\frac{-2}{p-1}}$, where $a = \frac{\alpha(p-1)^2}{2(p+1)}$ defines a nontrivial, global solution to (108). Thus the condition of compact support is necessary for the obstruction.

Remark 2.15. The exponent $1+\sqrt{2}$ is indeed critical here. In [5], John also shows that global solutions of $\Box u = |u|^p$, for $p > 1+\sqrt{2}$, exist for any initial data of compact support that are sufficiently small in a suitable norm.

Proof. (of Lemma 2.13) We sketch the argument as it is in [5].

Let u be a global solution satisfying the conditions of the lemma, and recall formula (7) which says $u = \underline{u} + \mathcal{L}(\Box(u))$, where

$$\mathcal{L}w = \frac{1}{4\pi} \int_0^t (t-s) \int_{|\eta|=1} w(x+(t-s)\eta, s) d\sigma(\eta) ds.$$

We note in particular that the Duhamel operator $\mathcal{L}$ obeys positivity

$$w \geq 0 \implies \mathcal{L}w \geq 0. \tag{114}$$

Let x_0 be as in the statement of the lemma and recall our notation of averaging around the point x_0

$$\tilde{w}(r,t) = \frac{1}{4\pi}\int_{|\xi|=1} w(x_0 + r\xi, t)d\sigma(\xi).$$

John observes the following calculation

$$\begin{aligned}(\widetilde{\mathcal{L}w})(r,t) &= \frac{1}{(4\pi)^2}\int_{|\xi|=1}\int_0^t (t-s)\int_{|\eta|=1} w(x+(t-s)\eta + r\xi, s)d\sigma(y)dsd\sigma(\xi) \\ &= \frac{1}{4\pi}\int_0^t \int_{|r-(t-s)|}^{r+(t-s)}\int_{|\xi|=1}\frac{\lambda}{2r}w(x_0+\lambda\xi, s)d\sigma(\xi)d\lambda ds \\ &= P\tilde{w}(r,t),\end{aligned} \tag{115}$$

where

$$Pv(r,t) = \iint_{R_{r,t}} \frac{\lambda}{2r} v(r,s)d\lambda ds$$

and

$$R_{r,t} = \{(\lambda, s)\colon t-r < s+\lambda < t+r, s-\lambda < t-r, 0 < s < t\}. \tag{116}$$

The change of variables in (115) may be visualised as the region foliated by space-time mantles which emanate from points a spacial distance of r from x_0, is parametrised as a surface of revolution, with cross-section given by $R_{r,t}$.

We note that P also obeys positivity,

$$v \geq 0 \implies Pv \geq 0. \tag{117}$$

Now, by (114) and the conditions of the lemma we have

$$u = \underline{u} + \mathcal{L}(\Box u) \geq \mathcal{L}(\alpha|u|^p). \tag{118}$$

at all points $(x,t) \in \Gamma(x_0, t_0)$. So, taking spherical averages gives by (117)

$$\tilde{u} \geq \alpha P(\widetilde{|u|^p}) \geq \alpha P(|\tilde{u}|^p), \tag{119}$$

where we applied Jensen's inequality giving $\widetilde{|u|^p} \geq |\tilde{u}|^p$. i.e. we have shown

$$\tilde{u}(r,t) \geq \alpha \iint_{R_{r,t}} \frac{\lambda}{2r}|\tilde{u}(\lambda,s)|^p d\lambda ds. \tag{120}$$

Assuming, for a contradiction, that there exists $(x_1,t_1) \notin C_0^{t_0}(x_0,t_0)$ such that $u(x_1,t_1) \neq 0$. Defining then $t_2 = t_1 + |x_1 - x_0|$, we have that

$$(x_0, t_2) \in \Gamma(x_0,t_0) \quad \text{and} \quad \in \mathcal{M}(x_0,t_0). \tag{121}$$

Since, $\mathcal{L}$ is given by an integral over the backward mantle, we deduce that

$$u = \underline{u} + \mathcal{L}(\Box u) \geq \alpha\mathcal{L}(|u|^p) > 0, \tag{122}$$

at the point (x_0, t_2). Thus we can find a $\delta > 0$ so small that $\tilde{u}(\lambda, s) \geq \varepsilon > 0$ for $(\lambda, s) \in [0, \frac{\delta}{2}) \times (t_2 - \frac{\delta}{2}, t_2 + \frac{\delta}{2})$.

It follows from (120) that

$$\tilde{u}(r,t) \geq \frac{c}{r}, \tag{123}$$

for some $c > 0$ whenever

$$(r,t) \in S := \{(\lambda, s) \colon t_2 + 2\delta \leq s + \lambda, t_2 \leq s - \lambda \leq t_2 + \delta\}. \tag{124}$$

John now employs a bootstrapping argument via the formula (120) to obtain successive lower bounds for $\tilde{u}(r,t)$. It is shown that for p in the critical range $(1, 1+\sqrt{2})$, these lower bounds will blow up for an appropriate value of (r,t). We sketch the argument here, and refer to [5] for the details.

Firstly is introduced the sets

$$\Sigma = \{(r,t) \colon 0 \leq r \leq t - t_2 - 2\delta\}, \tag{125}$$

$$S_{r,t} = \{(\lambda, s) \colon t - r < \lambda + s < t + r; t_2 < s - \lambda < t_2 + \delta\}. \tag{126}$$

Then applying (120), (123) gives, for $(r,t) \in \Sigma$

$$\begin{aligned} \tilde{u}(r,t) &\geq \alpha \iint_{R_{r,t}} \frac{\lambda}{2r} |\tilde{u}(\lambda, s)|^p d\lambda ds \\ &\geq \frac{\alpha c^p}{2r} \iint_{S_{r,t}} \lambda^{1-p} d\lambda ds, \end{aligned}$$

which is calculated as

$$\tilde{u}(r,t) \geq \delta \alpha c^p 2^{p-2} (t + r - t_2)^{1-p}. \tag{127}$$

Now, $(t + r - t_2)^{1-p} \geq (t+r)^{1-p}$ and, when $0 < p - 1 \leq 1$ we have

$$(t+r)^{p-1} = \left(\frac{t+r}{\delta}\right)^{p-1} \delta^{p-1} \leq (t+r)\delta^{p-2},$$

which implies $(t+r)^{1-p} \geq \delta^{2-p}(t+r)^{-1}$. It then follows that

$$\tilde{u}(r,t) \geq c_0 (r+t)^{-q_0}, \tag{128}$$

where $c_0, q_0 > 0$ do not depend on (r,t) and, in particular,

$$1 \leq q_0 = \begin{cases} 1 & \text{for } 1 < p \leq 2, \\ p - 1 & \text{for } 2 < p < 1 + \sqrt{2}. \end{cases} \tag{129}$$

For the bootstrap, assume now for some constants $C > 0$, $q \geq 1$, $a \geq 0$, $b \geq 0$, and for $\tau := t_2 + 2\delta$, there holds for $(r,t) \in \Sigma$ an inequality of the form

$$\tilde{u}(r,t) \geq C(t+r)^{-q}(t - r - \tau)^a (t-r)^{-b}. \tag{130}$$

Then defining the new set

$$T_{r,t} = \{(\lambda, s) \colon t - r < \lambda + s < t + r, \tau < s - \lambda < t - r\}, \tag{131}$$

by calculation one obtains from (120), (128), for $(r,t) \in \Sigma$,

$$\begin{aligned} \tilde{u}(r,t) &\geq \alpha \iint_{R_{r,t}} \frac{\lambda}{2r} |\tilde{u}(\lambda, s)|^p d\lambda ds \\ &\geq \frac{\alpha c^p}{2r} \iint_{T_{r,t}} \lambda (t+r)^{-pq} (t - r - \tau)^a (t-r)^{-b} d\lambda ds \\ &\geq \cdots \\ &\geq c^* (t+r)^{-1} (t - r - \tau)^{a^*} (t-r)^{-b^*}, \end{aligned} \tag{132}$$

where

$$a^* = pa + 2, \qquad b^* = p(b+q) - 1, \qquad c^* = \frac{\alpha c^p}{4(pa+2)} \min\left\{ \frac{1 - 2^{-pq}}{2(pq-1)}, 2^{1-pq} \right\}. \tag{133}$$

Applying (132) inductively and keeping track of these constants, John obtains

$$\tilde{u}(r,t) \geq \frac{t-r}{(t+r)(t-r-\tau)^{2/(p-1)}} \exp[p^k J(r,t)], \tag{134}$$

where $J(r,t) = E + \frac{2}{p-1} \log(t - r - \tau) - q_0 \log(t-r)$ for some constant E. Then, crucially, when $p < 1 + \sqrt{2}$, i.e. when $\frac{2}{p-1} > q_0$, we have that $\tilde{u}(r,t) = \infty$ whenever $t - r$ sufficiently large by passing to the limit as $k \to \infty$. This is the desired contradiction. □

The above result proves that blow-up is necessary for some compactly supported initial data, but does not give any insight into the blow up time. For the case of the equation $\Box u = u^2$, John proves the following theorem.

Theorem 2.16. *For given $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, let initial data be of the form $u_0 = \varepsilon\varphi$, $u_1 = \varepsilon\psi$. Letting $T^* = T^*(\varepsilon)$ denote the blow-up time, there exist positive constants A, B, ε_0 depending on φ and ψ but not on ε such that*

$$A\varepsilon^{-2} < T^* < B\varepsilon^{-2} \quad \textit{for } |\varepsilon| < \varepsilon_0.$$

Remark 2.17. This result was furthered by Kato in [6] to arbitrary space dimension m, to equations of the form

$$u_{tt} + Lu = f(t, x, u),$$

where L is an elliptic operator whose adjoint satisfies the condition $L^*(1) = 0$ and where the nonlinearity has growth

$$f(t,x,s) \geq \begin{cases} \alpha |s|_0^p, & |s| \leq 1, \\ \alpha |s|^p, & |s| > 1, \end{cases}$$

where $1 < p \leq p_0 = \frac{n+1}{n-1}$.

2.5 Further results

For the energy critical, defocussing equation,

$$\begin{cases} \Box u = -|u|^{2^*-2}u, \\ u(0) = u_0; \quad \partial_t u(0) = u_1, \end{cases} \tag{135}$$

Struwe's result was generalised by Grillakis [3] to give global existence for general, non-radially symmetric data when $n = 3$, and further by Grillakis [4] to give global existence for general data when $n = 4, 5$ and for radially symmetric data when $n = 6, 7$.

Moreover, work of Shatah-Struwe [10], [11] and of Bahouri-Shatah [1] showed higher regularity (scattering) of solutions. In particular, in $n = 3$ dimensions they proved $u \in L^4([0, \infty); L^{12}(\mathbb{R}^3))$ for finite energy solutions.

For the corresponding energy critical, *focussing* wave equation,

$$\begin{cases} \Box u = |u|^{2^*-2}u, \\ u(0) = u_0; \quad \partial_t u(0) = u_1, \end{cases} \tag{136}$$

where energy does not give a norm, Levine [9] showed that blow-up is exhibited in all cases where the initial energy is negative. i.e.

$$E_0 = E(u_0, u_1) = \int_{\mathbb{R}^n} \frac{1}{2}\left(|u_1|^2 + |\nabla u_0|^2\right) - \frac{1}{2^*}|u_0|^{2^*} dx < 0. \tag{137}$$

In the defocussing case, a stationary solution exists given by

$$\bar{W}(x, t) = W(x) = \left(1 + \frac{|x|}{n(n-2)}\right)^{-\frac{n-2}{2}} \tag{138}$$

with $W \in \dot{H}^1(\mathbb{R}^n)$. This stationary solution shows that finite energy solutions do not scatter in general. Kenig & Merle [8] showed that, in the case of $n = 3, 4, 5$ dimensions, this stationary element represents a critical element for both global existence and scattering in the following sense.

Theorem 2.18 (Kenig–Merle)**.** *If initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, $3 \le n \le 5$, and if $0 \le E(u_0, u_1) < E(W, 0)$, then:*

(i) if $\int_{\mathbb{R}^n} |\nabla u_0|^2 dx < \int_{\mathbb{R}^n} |\nabla W|^2 dx$, then solutions u to (136) exist for all time, and $\|u\|_{L^{2\frac{n+1}{n-2}}(\mathbb{R}^{n+1})} < \infty$,

(ii) if $\int_{\mathbb{R}^n} |\nabla u_0|^2 dx > \int_{\mathbb{R}^n} |\nabla W|^2 dx$, then solutions u to (136) blow-up in finite time.

A Hardy's inequality

Lemma A.1 (version 1)**.** *For all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ there holds*

$$\int_{\mathbb{R}^3} \frac{\varphi^2}{|x|^2} dx \le 4 \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx.$$

Proof. If $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, then by parts

$$\int_{-\infty}^{\infty} \psi^2 dr = \int_{-\infty}^{\infty} r^2 \psi^2 \frac{d}{dr}\left(\frac{-1}{r}\right) dr = 2\int_{-\infty}^{\infty} \psi^2 dr + 2\int_{-\infty}^{\infty} r\psi\psi' dr,$$

and applying Cauchy-Schwartz gives

$$\int_{-\infty}^{\infty} \psi^2 dr \leq 4\int_{-\infty}^{\infty} (\psi')^2 r^2 dr.$$

Thus, for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\varphi^2}{|x|^2} dx &= \int_{S^2}\int_0^\infty \varphi^2(ry) dr d\sigma(y) = \frac{1}{2}\int_{S^2}\int_{-\infty}^{\infty} \varphi^2(ry) dr d\sigma(y) \\ &\leq 2\int_{S^2}\int_{-\infty}^{\infty} (\partial_r \varphi(ry))^2 r^2 dr d\sigma(y) = 4\int_{\mathbb{R}^3} (\partial_r \varphi)^2 dx.\end{aligned}$$

□

Lemma A.2 (version 2)**.** *There exists an absolute constant C, independent of R, such that for all $\varphi \in \mathcal{C}^\infty(B_R)$*

$$\int_{B_R} \frac{\varphi^2}{|x|^2} dx \leq C\left(\int_{B_R} |\nabla\varphi|^2 dx + \left(\int_{B_R} \varphi^6 dx\right)^{\frac{1}{3}}\right).$$

Proof. Letting $\eta \equiv 1$ on $B_{R/2}$, $\eta \equiv 0$ near ∂B_R with $|\nabla\eta| \leq \frac{C}{R}$ we have by the previous lemma

$$\begin{aligned}\int_{B_R} \frac{\varphi^2}{|x|^2} &\leq \int_{B_R} \frac{(\varphi\eta)^2}{|x|^2} dx + \int_{B_R\setminus B_{R/2}} \frac{\varphi^2}{|x|^2} dx \\ &\leq 4\int_{B_R} |\nabla(\varphi\eta)|^2 dx + \frac{4}{R^2}\int_{B_R} \varphi^2 dx \\ &\leq C\left(\int_{B_R} |\nabla\varphi|^2 dx + \frac{1}{R^2}\int_{B_R} \varphi^2 dx\right) \\ &\leq C\left(\int_{B_R} |\nabla\varphi|^2 dx + \left(\int_{B_R} \varphi^6 dx\right)^{\frac{1}{3}}\right),\end{aligned} \tag{139}$$

where Hölder's inequality was applied in the last line. □

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