Initial Value and Initial-boundary Value Problems for Timelike Maximal Surfaces in (1+2)-Minkowski Space.

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy by

Edmund Adam Paxton.

The Queen's College, University of Oxford.

Supervised by

Prof. Luc Nguyen.

August 2019.

Acknowledgements

Firstly, I thank my supervisor, Luc Nguyen, who has dedicated so much of his time to act as a mentor to me. Thank you Luc for guiding me, for reading my work so carefully, for many insightful comments, and for being so patient. Thank you also to Jonny Evans who inspired me so much during the final year of my undergraduate degree and propelled me into this DPhil. And thanks to Sasha who set me up for my undergraduate degree years ago with those late nights doing maths at Queens.

I have benefitted from the teaching and support of many academics during the last 4 years. My special thanks to Melanie Rupflin, who devoted so much time to teaching me about harmonic maps and has always kept an open door. Thank you to Yves Capdeboscq who has been so encouraging towards me; to both Yves and Qian Wang, who buoyed me up with their kind words during my transfer; and to both Melanie and Endre Suli, who gave me yet another dose of positivity and support during my confirmation. My thanks to Mihalis Dafermos from whom it is inspirational to learn about Lorentzian geometry, and who has been so friendly towards us graduate students, even after we had dragged him up the bleak and desolate mountainside of Yobetsu during a conference in rainy Hokkaido last July. I was privileged to be invited to speak at the Yorkshire & Durham geometry day in May, and my thanks go to the organisers as well as everyone who was present at that day who made it such a special experience for me. I am much indebted to Jens Hoppe who has readily engaged with me over skype and pointed me to many interesting references, and who has even invited me to Paris as his guest. Thank you also to Jens Eggers who hosted me in Bristol and spoke with me at length about my work; to Jason Lotay who has shown so much interest in my work, who has sought to encourage me, and who is invariably charged with exciting new ideas; to Chris Hollings for some delightful references on the history of geometry; to Andrei Starinets for an amazing account of holography; and to Andreas Braun for being a good friend and for his enthusiasm when trying to explain string theory to me.

My thanks to the organisers of the CDT, and to the EPSRC who funded everything and thus made it all possible. It has been great to be part of a CDT with so many fellow graduate students, and my thanks go to the guys from my cohort with whom I have formed many close friendships. Thanks also to Candy, Kelli and Luci who I am so lucky to have met through the CCG; to Sandy for the administrative support and guidance; to Hamza, who helped me with Python; to the guys who shared my office in the last year, Avi, Joe and Tristan, who have been awesome (Tristan that includes your help with English, and Joe your willingness to procrastinate on some obscure geometry); and to Matthew Schrecker, who carefully read through my introduction.

I simply would not be where I am today were it not for my family. First and foremost, my parents. To my dad, Tony, who has shared with me his wonderful knowledge of the physical world and who has always cheered me on with characteristic enthusiasm, and to my mum, Sarah, who always wanted me to grow up free, and who has sustained me during my studies with countless goodie bags packed with home-cooked food and with infinitely many more acts of love, I thank you both so much and this thesis is dedicated to you. Thanks to Zoë for always looking out for me and for the healthy doses of realism, and to Alfie for taking me away from my work for an awesome holiday in Vietnam (and for much more, of course). When I look back on my studies, I know that one of my best memories will be of sitting by the fireplace in the peace and quiet of Old Windsor, doing some maths, with Grandmüs preparing a hot meal in the next room. It's amazing Grandmüs that you are still going so strong at almost 99 years old and I feel so lucky to have had so much of you. So too will I have fond memories of Granny Do who would put me up to stay in her flat in Manchester and dote upon me, and who right up until her passing away this year would speak to me on the phone and would tell me about the nature outside her window, and that everything will work out fine. My thanks to my excellent Aunts, Uncles and Cousins: Ann, Paul, Ilan, Yuval, Robin, Linda, Conrad and Caspar, who have always been really encouraging. In particular thank you to Paul who had done a maths DPhil and who was there to support me on a number of occasions. And thanks to my godparents Adrian and Pat who were standing by in Oxford ready to come to the rescue when necessary, including on one occasion when we were forced from our home by all-night pneumatic drilling.

I have always been lucky to have a great network of friends to turn to. Thank you in particular to James and Duncan who have been there since we met at the UCL juggling soc in our first year of university and who are truly excellent friends. Thanks to Graham, Joe, and Liam who have also been there since the UCL days and with whom it has been really special to be able to share our respective maths PhD experiences. And to everyone from Belfast, there are too many of you to list but you know who you are and you have been with me through it all, from pokemon and 40:40 to pints in the Bosnian hills, so thanks for all the great craic. Finally, thank you to Cassy, who has accompanied me on many adventures, and who has helped me throughout to keep on enjoying maths by taking me out into the wild, and reminding me to turn my face to the sun, to feel the warmth of its rays, to breathe, and to look around at a beautiful world.

Contents

List of Figures

Summary of the thesis

This thesis investigates timelike maximal surfaces in the Minkowski space $\mathbb{R}^{1+2}$, from the perspective of both the initial value problem (IVP) and initial-boundary value problems (IBVPs).

Timelike maximal surfaces have been proposed for a number of physical theories (see §1.1.3) and their dynamics give rise to an interesting case study for a geometric evolution of wave type. A thorough review of the literature on timelike maximal surfaces will be found in §1.3, but for the purposes of this summary let us briefly mention some key points. The global dynamics of spatially compact timelike maximal surfaces in $\mathbb{R}^{1+2}$ (surfaces diffeomorphic to $S^1 \times \mathbb{R}$) are relatively well understood. In particular, it is known that solutions to the IVP for such surfaces (i.e. the flow of closed curves) always become singular in finite time. Moreover, the dynamics of spatially non-compact timelike maximal surfaces in $\mathbb{R}^{1+2}$ (surfaces diffeomorphic to $\mathbb{R}^2$) within the regime of 'small-data' are also relatively well-understood. Here it is known that the Cauchy evolution of a smooth non-compact curve of 'small' curvature and velocity in $\mathbb{R}^{1+2}$ yields a global solution to the IVP which is a smooth properly embedded graphical timelike maximal surface in $\mathbb{R}^{1+2}$ 'close' to a flat timelike plane. Many illuminating results have been obtained regarding the mechanisms for singularity formation for timelike maximal surfaces in $\mathbb{R}^{1+2}$, but there still remain interesting questions to be answered. For example, we do not have a satisfactory classification of all possible mechanisms for singularity formation. Moreover, it is not

known whether there exists a 'good' (in particular, C^1 and unique) notion of timelike maximal surface to be adopted after a singularity has formed (i.e. a good notion of weak solution) in the generic case of singularity formation. Finally, whilst IBVPs for timelike maximal surfaces arise in certain physical theories, the author does not know of mathematical treatments of such problems aside from some work concerning global existence results within the small-data regime.

Loosely speaking, the main aims of this thesis are as follows: (1) to 'close the gap' between the spatially compact and spatially non-compact regimes, by investigating in more detail the dynamics of spatially non-compact timelike maximal surfaces in $\mathbb{R}^{1+2}$ (in the 'large-data' regime); (2) to further investigate singularity formation for timelike maximal surfaces in $\mathbb{R}^{1+2}$; (3) to develop a framework to solve IBVPs for timelike maximal surfaces in $\mathbb{R}^{1+2}$ by conformal methods and to study the conformal structures of solutions to such IBVPs.

We will now briefly survey the main results of this thesis (refer to §1.4 for a more detailed presentation). We prove that

(I) if $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ is a smooth proper timelike immersion with vanishing mean curvature then ϕ is an embedding and every compact subset of $\mathrm{Im}(\phi)$ is a smooth graph (Theorem 2.3).

We also construct examples of smooth proper timelike maximal immersions $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ for which $\mathrm{Im}(\phi)$ is not a graph, demonstrating that the restriction to compact subsets in (I) cannot be relaxed. These may be viewed as 'large-data' results, and (I) has consequences for the IVP for timelike maximal surfaces (i.e. *given a smooth proper spacelike planar immersion* $C\colon \mathbb{R} \to \mathbb{R}^{1+2}$ *and a smooth timelike vector field* V *along* C*, find a smooth timelike maximal surface in* $\mathbb{R}^{1+2}$ *which contains* $\mathrm{Im}(C)$ *and is tangent to* V *along* C*).* Indeed, it may be seen to follow from (I) that if the image of the unit tangent vector U_0 along C contains a closed semi-circle (or, equivalently, if

$\mathrm{Im}(C)$ contains a compact subset which is not a smooth graph) then any inextendible solution to the IVP must become singular, at some point in either the future or the past. We prove that

(II) singularity formation always involves a blow-up of spatial curvature, and the curvature blow-up occurs in an $L^1_{\mathrm{time}}L^\infty_{\mathrm{space}}$-norm (Theorem 3.1).

This result provides a further step towards a classification of singularity formation. It follows from (II) that a solution to the IVP which becomes singular in finite time will be C^2 inextendible beyond the singular time. We analyse in some detail the method of evolution by isothermal gauge, which gives a well-known notion of evolution beyond singular time. We show that there exist (non-generic) examples of smooth initial data (C, V) for which $\mathrm{Im}(U_0)$ is exactly a closed semi-circle and for which the spatial unit tangent along the evolution by isothermal gauge of (C, V) extends continuously to a globally defined unit tangent vector field. In these examples, we then deduce that the evolution by isothermal gauge defines a global C^1 extension beyond singular time. But we also show that this situation is 'borderline'. To be precise, we prove that

(III) if $\mathrm{Im}(U_0)$ contains an arc of length $> \pi$ then the spatial unit tangent along the evolution by isothermal gauge admits no continuous extension to a unit tangent vector field (Theorem 4.16).

This result shows that, outside of the 'borderline' cases, the evolution by isothermal gauge is never a 'good' notion of global solution.

In addition to the above results, we develop a framework for analysing IBVPs for timelike maximal surfaces by conformal methods. The boundary conditions that we consider are: (a) a single (arbitrary) timelike curve, (b) a single timelike plane, and (c) a pair of parallel timelike lines. In each of the cases (a), (b) & (c) we show how to solve the IBVP globally in isothermal gauge, yielding a global solution with possible singular points (i.e. we show how to construct a Weierstrass-type formula in each case). For the case of a non-empty singular set we present some analysis of the singular points. We also derive some estimates on the initial-boundary data which imply an empty singular set, thus yielding global existence of solutions to the IBVP for certain non-empty open sets of initial-boundary data. These are the first such non-perturbative global existence results. For case (b) our methods could be applied to a general timelike boundary surface, but we restrict attention to the plane to keep the presentation simple. Our methods do not give explicit representation formulas except for in a few special cases (a single timelike line, a single timelike plane, a pair of parallel timelike lines) which we treat in detail. A final point should be mentioned concerning our method. To apply our method to an IBVP, we must first choose a priori the conformal structure of the solution. A weakness of this approach is that we must impose additional C^2 compatibility conditions on the initial-boundary data 'at the corner' in order to obtain existence (see (5.15)). A strength of the method is that we do obtain global solutions with our choice of conformal domain—which has a particularly simple structure of null infinity—and we are then able to infer that the global solution to the IBVP has a simple null infinity a postiori.

Note for the reader: Refer to Appendix A for a detailed account of the structure of the thesis, and refer to Appendix B for the notation and terminology of the thesis. Most of the material from Chapters 2–4 may be found in the author's preprint [65] (submitted for publication).

Chapter 1

An invitation to timelike maximal surfaces

1.1 Historical context & motivation

1.1.1 A brief history of minimal surfaces in Euclidean space and the conformal method.

In 1760, Lagrange [47] wrote down the equation

$$\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right)_x + \left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right)_y = 0 \tag{1.1}$$

describing a graphical surface $\Sigma = \{(x, y, u(x, y))\} \subseteq \mathbb{R}^3$ which extremizes the area functional with respect to compactly supported variations (what is now called a minimal surface). The equation (1.1) is nonlinear and Lagrange did not give explicit solutions apart from the flat planes $D^2u \equiv 0$. In 1776, Meusnier [55] identified (1.1) as being equivalent to the condition that the surface Σ has vanishing mean curvature and gave examples of non-trivial solutions to (1.1), the catenoid and the helicoid (neither of which can be written globally as a graph, but which solve (1.1) over local graphical patches).

Of special importance in the analysis of (1.1) has been the development of conformal methods. If $\Sigma \subseteq \mathbb{R}^3$ is a smooth minimal surface then there are of course many different ways to parameterise Σ, and since the mean curvature is independent of the parameterisation, one may then facilitate the solution of the maximal surface equation (thanks to Meusnier's observation) by exploiting this freedom. Indeed, if $\phi\colon U^2 \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth immersion then the mean curvature is

$$H(\phi) = \frac{1}{\sqrt{\det(g)}} \partial_i \left(\sqrt{\det(g)} g^{ij} \partial_j \phi \right) \tag{1.2}$$

where $g_{ij} = \langle \partial_i \phi, \partial_j \phi \rangle$ is the first fundamental form and we have adopted the summation convention with $g^{ij} g_{jk} = \delta^i_k$. Now suppose that ϕ is also a conformal map with respect to the Euclidean metric $ds^2 = dx^2 + dy^2$ on $\mathbb{R}^2$ i.e. suppose the conditions

$$|\phi_x(x,y)|^2 - |\phi_y(x,y)|^2 = 0 \tag{1.3}$$

$$\langle \phi_x(x,y), \phi_y(x,y) \rangle = 0 \tag{1.4}$$

hold. Then the equation $H(\phi) = 0$ becomes the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0. \tag{1.5}$$

Writing the conditions (1.3) and (1.4) in complex coordinates as $|\phi_x + i\phi_y|^2 = 0$ and using the fact that the real and imaginary parts of any holomorphic function on the complex plane satisfy (1.5), it turns out to be possible to construct conformal parametrizations of minimal surfaces from holomorphic functions by algebraic means. This is achieved via the elegant Weierstrass representation formula for minimal surfaces [63, Lemma 8.1].

Over the past two and a half centuries, many interesting examples of minimal surfaces in $\mathbb{R}^3$ have been constructed. These include examples of complete minimal surfaces in $\mathbb{R}^3$ of a variety of topological types (e.g. the surfaces of Scherk) as well as examples of complete minimal surfaces in $\mathbb{R}^3$ with self-intersections (e.g. the surfaces of Enneper, Figure 1.1). On the other hand, many remarkable theorems have demonstrated that the complete minimal surfaces in $\mathbb{R}^3$ exhibit a certain rigidity. Let us mention here a few such results. Bernstein famously proved that any complete minimal surface in $\mathbb{R}^3$ which is a graph, must be a plane [11]. Osserman extended this result and proved that for any complete minimal surface in $\mathbb{R}^3$ the image of the unit normal vector is either a single point, in the case of a plane, or it is dense in the unit sphere S^2 [61]. Osserman's theorem was sharpened over some years until Fujimoto proved that either the image of the unit normal is a single point, or it omits at most 4 points in the unit sphere [30]. This is the sharpest possible bound since Scherk surfaces have a unit normal which omits exactly 4 points in the unit sphere [62]. A survey of these results (excluding Fujimoto's theorem) and much more can be found in Osserman's book [63].

1.1.2 Der glücklichste Gedanke meines Lebens: the advent of Lorentzian geometry

The subject of Lorentzian geometry was developed at the turn of the 20^{th} century by Einstein, Grossman, Lorentz, Minkowski, and others, and emerged at the forefront of mathematical research due to the success of the theory of relativity [67, pp.35–38], [31, Chap.16–17].

To illustrate the central role that Lorentzian geometry plays in modern physics, let us consider an English cricketer striking the ball for 6 in this year's recent world

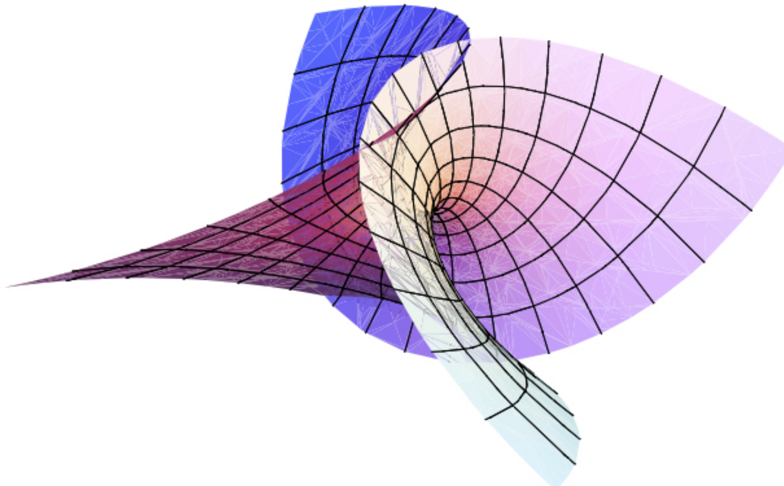


Figure 1.1: Enneper's minimal surface. A complete self-intersecting minimal surface in $\mathbb{R}^3$ whose unit normal omits exactly one point on the sphere (image obtained with permission from MathWorld [71])

.

cup final. The student of the 19$^{\text{th}}$ century would have been taught the following picture. The ball is struck into the air and acted upon by a force due to gravity. This force acts exactly in proportion to the ball's inertial mass and thus induces the ball to accelerate at a constant rate ($\approx$9.8ms^{-2}) towards the earth. The ball traverses a perfect parabola as it passes the boundary and returns to earth somewhere in the crowd. For the present day undergraduate, however, this picture is turned on its head. The ball now feels no force at all, but instead traverses a path of least action in space-time (no acceleration). It is in fact the cricketer who feels a force beneath his feet as he, and the entire of Lord's cricket ground, are accelerated upwards towards the ball! Far more than an exercise in semantics, this wonderful new point of view is crucial to the theory of relativity, and was referred to by Einstein as the "happiest thought of my life" [26], [64, p178].

The example discussed above is supposed to illustrate a radical modern perspective on the world, and the mathematical structure behind this viewpoint (the thing which gives meaning to the question of whether it is the cricketer or the ball which is accelerated) is a Lorentzian manifold. We are now taught in undergraduate courses that physical laws may be expressed in terms of a Lorentzian metric and the field of Lorentzian geometry, together with the geometric partial differential equations associated with it, would appear to provide the mathematical canvas on which our vibrant picture of reality is painted. For the perfect discussion of space and time and of what it means to be at rest, we refer to Einstein's book [27].

1.1.3 Timelike maximal surfaces

Since the arrival of the theory of relativity, many physical theories have been formulated within the framework of Lorentzian geometry. Born & Infeld [15] and later Dirac [21] were among prominent physicists of the 20$^{\text{th}}$ century who proposed new theories of electromagnetism. In one simplified setting, the equations of the Born-Infeld theory for a plane electromagnetic wave reduce to

$$\left(\frac{u_t}{\sqrt{1+u_x^2-u_t^2}}\right)_t - \left(\frac{u_x}{\sqrt{1+u_x^2-u_t^2}}\right)_x = 0, \tag{1.6}$$

see Barbashov & Chernikov [6]. Equation (1.6) is often referred to as the Born-Infeld equation and it is the hyperbolic analogue of Lagrange's minimal surface equation (1.1). Analogously to the situation for the minimal surface equation, (1.6) describes a timelike surface $\Sigma = \{(t, x, u(x,t))\}$ in Minkowski space $\mathbb{R}^{1+2}$ with vanishing mean-curvature, which is called a timelike maximal surface.

Timelike maximal surfaces have many properties giving them appeal for physical theories: the mean curvature is a geometric object, in that it is independent of the choice of local coordinates on the surface and invariant under isometries of the ambient space; timelike maximal surfaces are extremal points of an area functional, so they satisfy a principle of least action; and the timelike maximal surface equations are invariant under rescalings of the ambient space, so their dynamics are independent of the choice of scale. In addition to the Born-Infeld theory, timelike maximal surfaces have arisen in numerous physical theories including in cosmological models of the early universe (as so-called cosmic strings [44]) and in bosonic string theory (known as extremal points of the Nambu-Goto action [43, Chap. 2]) where they have enjoyed some particular attention recently due to the successes of holography in certain computations of particle physics [54], [18, Section 6].

Aside from all potential applications, given the beauty of the theory of minimal surfaces in $\mathbb{R}^3$, it seems natural to study the timelike maximal surfaces in $\mathbb{R}^{1+2}$ in their own right as a subdiscipline of Lorentzian geometry. This is the standpoint from which this thesis is written, and for the rest of this document we will be motivated by purely mathematical considerations.

1.2 Timelike maximal surfaces: an evolution problem

1.2.1 The initial value problem (IVP)

We now turn to the topic of this thesis, timelike maximal surfaces. Since we are to study a geometric PDE, it is important for us to specify a notion of global solution. A good notion of global solution in the context of timelike maximal surfaces is given by a properly immersed surface. This is illustrated by the following lemma, which shows us that a properly immersed timelike surface in $\mathbb{R}^{1+2}$ may be visualised as an unbroken flow of immersed planar curves.

Lemma 1.1. *Suppose that U^2 is a smooth connected surface and $\phi\colon U^2 \to \mathbb{R}^{1+2}$ is a smooth proper (i.e. the preimage of any compact set is compact) timelike immersion.*

Then there exists a smooth connected 1-manifold Λ^1 *(either* $\Lambda^1 = S^1$ *or* $\Lambda^1 = \mathbb{R}$*) and a smooth diffeomorphism* $\psi \colon \Lambda^1 \times \mathbb{R} \to U^2$ *such that*

$$(\phi \circ \psi)(s,t) = (t, \gamma(s,t))$$

for some smooth $\gamma \colon \Lambda^1 \times \mathbb{R} \to \mathbb{R}^2$ *where each immersion* $\gamma(\cdot, t) \colon \Lambda^1 \to \mathbb{R}^2$ *is proper.*

Proof. The proof is by Morse theory and may be found in [4]. Alternatively, refer to our proof of Lemma 2.1 in §2.1 in which the main ideas may be found. □

From Lemma 1.1 we see that there are only two possible topologies for a properly immersed timelike surface, the spatially compact case $S^1 \times \mathbb{R}$ and the spatially non-compact case $\mathbb{R}^2$. Moreover, Lemma 1.1 shows that the x^0 coordinate on $\mathbb{R}^{1+2}$ restricts to give a natural time-function on a properly immersed timelike surface. We are now in a good position to state the initial value problem (IVP) for timelike maximal immersions.

Definition 1.2 (The initial value problem (IVP) for timelike maximal immersions)**.** Let $\Lambda^1 = S^1$ or $\mathbb{R}$ and $C \colon \Lambda^1 \to \mathbb{R}^{1+2}$ be a smooth proper immersion of the form $C(s) = (0, c(s))$ and let V be a smooth future-directed timelike vector field along C. We call (C, V) an initial data. Given an initial data (C, V), the IVP is to find a $T \in (0, \infty]$ and a smooth proper timelike maximal[1] immersion $\phi \colon \Lambda^1 \times [-T, T] \to \mathbb{R}^{1+2}$ in the case $T < \infty$ or $\phi \colon \Lambda^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ in the case $T = \infty$ of the form $\phi(s,t) = (t, \gamma(s,t))$ such that $s \mapsto \phi(s, 0)$ is a monotone reparameterisation of C and V is tangent to $\mathrm{Im}(\phi)$ along C. If $T < \infty$ we say that ϕ is a local solution to the IVP and if $T = \infty$ we say that ϕ is a global solution to the IVP.

Remark 1.3. We may also refer to a solution to the IVP for an initial data (C, V) as a Cauchy evolution of (C, V). We may refer to $\phi\big|_{t \geq 0}$ as a future Cauchy evolution

[1] i.e. vanishing mean curvature, see Appendix B

of (C, V) and $\phi\big|_{t\leq 0}$ as a past Cauchy evolution of (C, V). Note that, for us, a global solution is an evolution of (C, V) towards both future and past.

1.2.2 The issue of gauge

Solutions to the IVP, as it is stated in Definition 1.2, will not be unique. This follows from the independence of the mean curvature from the system of coordinates. Indeed, suppose that $\phi\colon \Lambda^1 \times [-T, T] \to \mathbb{R}^{1+2}$ is a solution to the IVP for an initial data (C, V) and let $\psi\colon \Lambda^1 \times [-T, T] \to \Lambda^1 \times [-T, T]$ be any smooth diffeomorphism of the form $\psi(s, t) = (\omega(s, t), t)$ where $s \mapsto \omega(s, t)$ is a monotone reparameterisation for all $t \in [-T, T]$. Then since the mean curvature satisfies $H(\phi) = H(\phi \circ \psi)$ it follows that $\phi \circ \psi$ will also be a solution to the IVP for the initial data (C, V).

To obtain a well-posed system of equations in order to analyse the IVP, one approach is to break this diffeomorphism invariance by fixing a priori some preferred choice of parameterisation (called fixing the gauge). To illustrate the idea of gauge fixing, suppose that we restrict ourselves to graphical parameterisations of the form $\phi\colon \mathbb{R} \times [-T, T] \to \mathbb{R}^{1+2}$;

$$\phi(s, t) = (t, s, u(s, t)). \tag{1.7}$$

Then the timelike maximal surface equation (B.1) reduces to

$$\left(\frac{u_t}{\sqrt{1 + u_x^2 - u_t^2}}\right)_t - \left(\frac{u_x}{\sqrt{1 + u_x^2 - u_t^2}}\right)_x = 0$$

which we recognise as the Born-Infeld equation (1.6). The Born-Infeld equation is strictly hyperbolic and thus admits a unique smooth solution $u\colon \mathbb{R} \times [-T, T] \to \mathbb{R}$ for some $T > 0$ given any smooth initial data $(u\big|_{t=0}, u_t\big|_{t=0})$ which decays sufficiently rapidly at infinity (see eg. [68, Theorem 5.1]). Thus by fixing the graphical gauge we obtain a well posed equation to analyse in (1.6).

Remark 1.4. One can only hope to obtain global solutions via the ansatz (1.7) provided the evolution of graphical data remains graphical for all time. More generally, the trick of gauge fixing will only work so long as the gauge conditions are 'propagated' by the timelike maximal surface equations.

1.2.3 The isothermal gauge

In the previous section we discussed the trick of fixing a gauge and it is now a good time to introduce the isothermal gauge. This will provide us with a host of examples of timelike maximal surfaces in $\mathbb{R}^{1+2}$, similarly to the way that the Weierstrass representation formula provides many examples of minimal surfaces in $\mathbb{R}^3$.

To understand why the isothermal gauge trick works, it is helpful for us first to understand a key property of timelike maximal surfaces. Suppose that $\Sigma \subseteq \mathbb{R}^{1+2}$ is a timelike maximal surface and let (z_+, z_-) be local null coordinates on Σ. That is, the local parameterisation of Σ by $(z_+, z_-) \mapsto \phi(z_+, z_-)$ is such that the vectors $\frac{\partial \phi}{\partial z_+}$ and $\frac{\partial \phi}{\partial z_-}$ are null. Then the timelike maximal surface equation (see (B.1)) reduces to the wave equation

$$\frac{\partial^2 \phi}{\partial z_+ \partial z_-} = 0. \tag{1.8}$$

It follows from (1.8) that the vectors $\frac{\partial \phi}{\partial z_+}$ are constant along lines $\{z_+ = \text{constant}\}$ whilst the vectors $\frac{\partial \phi}{\partial z_-}$ are constant along lines $\{z_- = \text{constant}\}$. We have arrived at a key property of timelike maximal surfaces.

Property 1.5. *On a timelike maximal surface, the outgoing null tangent directions are constant along the integral curves of the incoming null vector fields, whilst the incoming null tangent directions are constant along the integral curves of the outgoing* *null vector fields.*

Property 1.5 motivates the following idea: choose a gauge for which the coordinate directions are null, and these gauge conditions will be 'propagated' by the equation. The following lemma shows that this idea works a treat.

Lemma 1.6 (Propagation of isothermal gauge). *Let $\Lambda^1 = S^1$ or $\mathbb{R}$ and suppose $\phi\colon \Lambda^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ is a smooth map which satisfies the wave equation*

$$\frac{\partial^2 \phi}{\partial z_+ \partial z_-} = \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial s^2} = 0 \tag{WE}$$

and which satisfies

$$\|\frac{\partial \phi}{\partial z_+}\|^2 = \|\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial s}\|^2 = 0 \tag{G1}$$

$$\|\frac{\partial \phi}{\partial z_-}\|^2 = \|\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial s}\|^2 = 0 \tag{G2}$$

along $\Lambda^1 \times \{0\}$ where $\|\cdot\|^2$ denotes the Minkowskian product (see Appendix B). Then ϕ satisfies (G1) *and* (G2) *on all of $\Lambda^1 \times \mathbb{R}$.*

Proof. From (WE) we have $\frac{\partial}{\partial z_+}\|\frac{\partial \phi}{\partial z_-}\|^2 = \frac{\partial}{\partial z_-}\|\frac{\partial \phi}{\partial z_+}\|^2 = 0$ and the lemma follows immediately. □

In light of Lemma 1.6, we now make the following definition.

Definition 1.7. Let (C, V) be a smooth initial data as in Definition 1.2. By reparameterising, without loss of generality we may take (C, V) of the form $C(s) = (0, c(s))$, $V(s) = (1, v(s))$ where $|v(s) + c'(s)|^2 = |v(s) - c'(s)|^2 = 1$ i.e. such that

$$A_\pm(s) = V(s) \pm C'(s) = (1, v(s) \pm c'(s)) = (1, a_\pm(s)) \tag{1.9}$$

are null. Let $\phi\colon \Lambda^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ denote the unique smooth solution to (WE) with initial conditions $\phi(s,0) = C(s)$, $\phi_t(s,0) = V(s)$. We call ϕ the evolution of (C,V) by isothermal gauge.

It may be seen that the evolution by isothermal gauge $\phi\colon \Lambda^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ is of the form

$$\phi(s,t) = (t, \gamma(s,t))$$

and from Lemma 1.6 it follows immediately that ϕ is a smooth timelike maximal immersion on $(\Lambda^1 \times \mathbb{R}) \setminus \mathcal{K}$ where

$$\mathcal{K} = \{(s,t) \in \Lambda^1 \times \mathbb{R}\colon \gamma_s(s,t) = 0\}.$$

If $\mathcal{K} = \emptyset$ (i.e. if ϕ is an immersion) then ϕ gives a global solution to the IVP of Definition 1.2. In general, however, ϕ will not be an immersion and ϕ will only give a local solution to the IVP.

Remark 1.8. Recall that solutions of the wave equation (WE) may be given explicitly in terms of the initial data by d'Alembert's formula

$$\gamma(s,t) = \frac{1}{2}\left(c(s+t) + c(s-t) + \int_{s-t}^{s+t} v(\zeta)d\zeta\right)$$

which implies

$$\begin{aligned}\gamma_s(s,t) &= \frac{1}{2}\left(c'(s+t) + c'(s-t) + v(s+t) - v(s-t)\right)\\ &= \frac{1}{2}\left(a_+(s+t) - a_-(s-t)\right)\end{aligned}$$

and we end up at the following equivalent characterisation of the singular set

$$\mathcal{K} = \{(s,t) \in \Lambda^1 \times \mathbb{R} \colon a_+(s+t) = a_-(s-t)\}.$$

In other words, ϕ is singular iff the images of the null directions along the initial data $\mathrm{Im}(a_+)$ and $\mathrm{Im}(a_-)$ overlap (compare with Property 1.5).

Let us conclude this section with two simple examples of timelike maximal surfaces obtained by isothermal gauge.

Example 1.9 (The timelike plane)**.** Let $C(s) = (0, s, 0)$ and $V(s) = (1, 0, 0)$. Then the evolution by isothermal gauge of (C, V) is given by $\phi \colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$, $\phi(s,t) = (t, s, 0)$. We have $\mathcal{K} = \emptyset$ (i.e. ϕ is an immersion) and $\mathrm{Im}(\phi)$ is the $\{x^2 = 0\}$ plane.

Example 1.10 (The shrinking circle)**.** Let $C(s) = (0, \cos s, \sin s)$ and $V(s) = (1, 0, 0)$. Then the evolution by isothermal gauge of (C, V) is given by

$$\phi(s,t) = (t, \cos t \cos s, \cos t \sin s).$$

We see that

$$\mathcal{K} = \left\{(s,t) \in S^1 \times \mathbb{R} \colon t \in \{\ldots, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}\right\}.$$

Viewing $\phi\Big|_{S^1 \times [0, \frac{\pi}{2})}$ as a future Cauchy evolution of planar curves (i.e. looking at the $\{x^0 = t\}$ cross sections of $\phi(S^1 \times [0, \frac{\pi}{2})$ as t ranges from 0 to $\frac{\pi}{2}$) we observe a family of circles which shrink to a point singularity in finite time, see Figure 1.2.

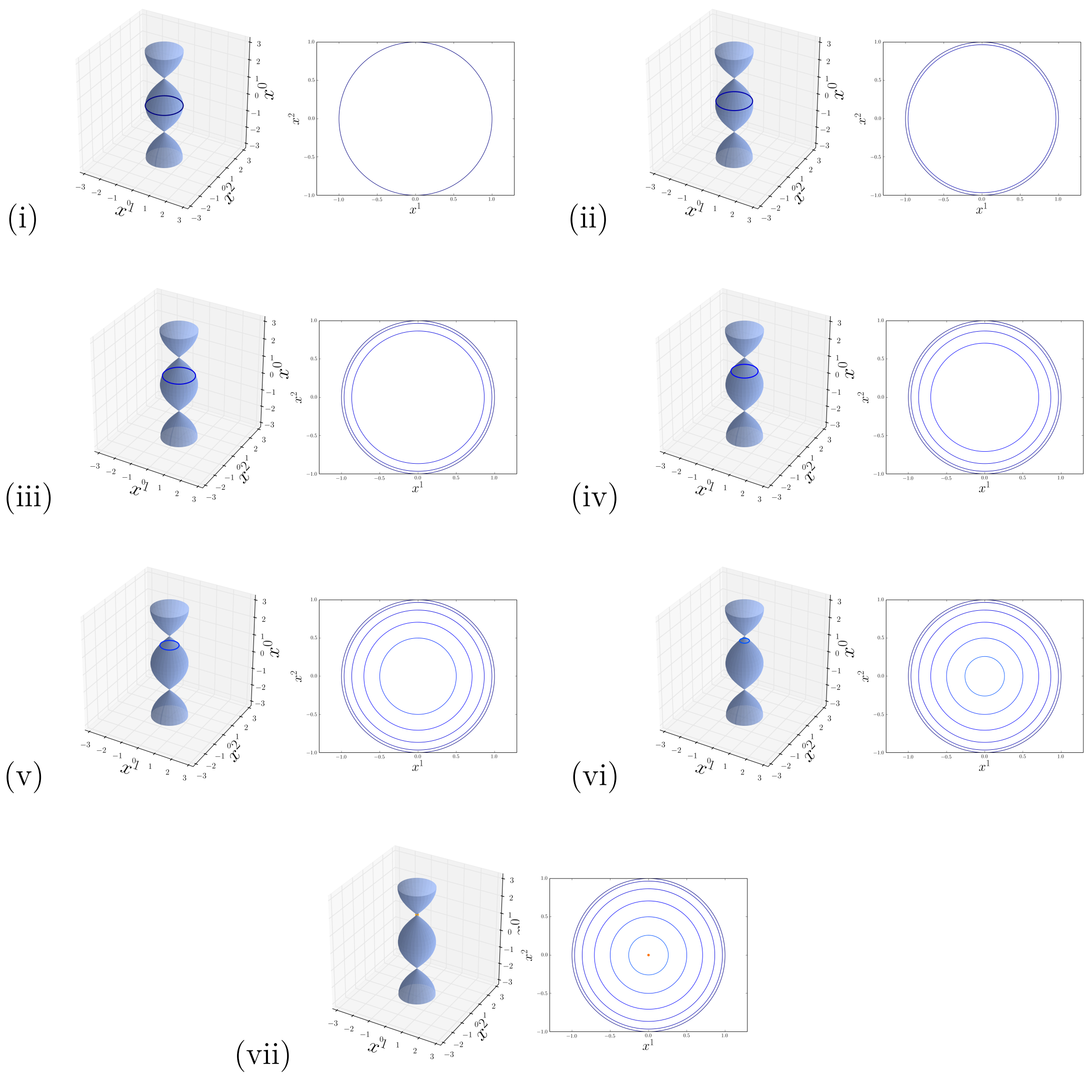


Figure 1.2: The shrinking circle solution of Example 3.3 plotted alongside its spatial cross sections $\{x^0 = t\}$ (projected onto the x^1–x^2 plane) for (i) $t = 0$, (ii) $t = \frac{\pi}{12}$, (iii) $t = \frac{2\pi}{12}$, (iv) $t = \frac{3\pi}{12}$, (v) $t = \frac{4\pi}{12}$, (vi) $t = \frac{5\pi}{12}$, (vii) $t = \frac{6\pi}{12} = \frac{\pi}{2}$. This is a future Cauchy evolution consisting of a family of circles which collapse to a point singularity in finite time $\frac{\pi}{2}$.

1.3 Literature review

1.3.1 Singularity results

As we saw for the shrinking circle (Example 1.10) solutions to the IVP for timelike maximal surfaces may become singular in finite time. In fact, given any closed initial curve $C\colon S^1 \to \mathbb{R}^{1+2}$ and any initial velocity V along C, it is known that any Cauchy evolution of (C, V) to a timelike maximal surface in $\mathbb{R}^{1+2}$ must become singular in finite time. Indeed, by solving the equations in isothermal gauge, Hoppe obtained a formula for the curvature of an evolving closed string and showed that it always blows up in finite time [36] (this was also noted in earlier work of Pronko, Razumov & Soloviev [66]). Nguyen & Tian proved the statement: there exists no smooth proper timelike immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ with vanishing mean curvature [59, Theorem 1.1].

Let us now discuss the literature on formation of singularities for timelike maximal surfaces in $\mathbb{R}^{1+2}$. The shrinking circle of Example 1.10 illustrates a special case of a theorem of Belletini, Hoppe, Novaga & Orlandi, who proved that given any smooth closed convex and centrally symmetric curve C with timelike velocity $V = (1, 0, 0) = \partial_{x^0}$ along C, the evolution of (C, V) by isothermal gauge consists of a family of smooth convex curves which shrink to a point in finite time [8, Propositions 5.2 & 5.4]. However, it may be seen that the collapsing solutions of [8] are unstable under perturbation and in general a closed convex curve (without symmetry assumption) will form a singularity in finite time before collapse. Let us proceed to discuss the generic case of singularity formation. Eggers & Hoppe [23] studied singularity formation for the Born-Infeld equation (recall (1.6)) under a self-similar ansatz in all spatial dimensions. They also compared their results with the evolution by isothermal gauge (Definition 1.7) which is available in the case of one space dimension. Recall

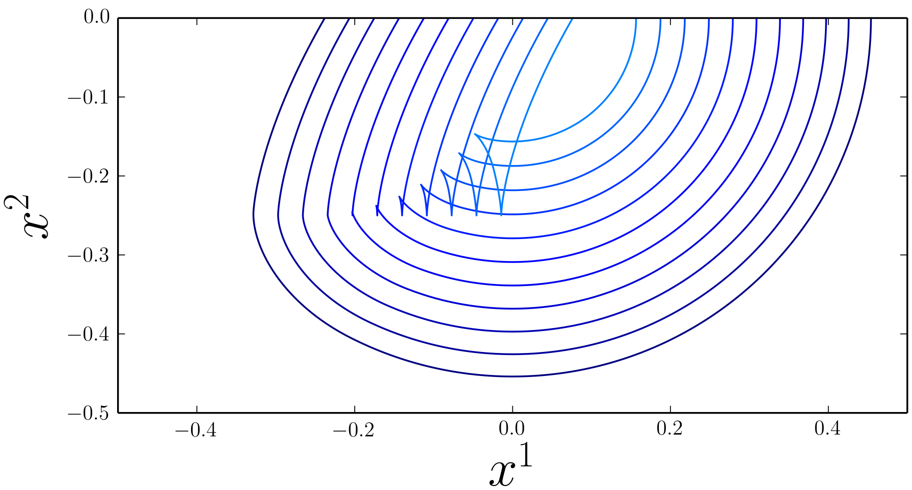


Figure 1.3: The swallowtail singularity. The spatial cross sections of the evolution by isothermal gauge for some initial curve C and some initial velocity V along C are plotted (projected onto the x^1–x^2 plane). The curve $C = \{x^0 = 0\}$ is plotted in darkest blue and the successively lighter blue curves represent spatial cross sections $\{x^0 = t\}$ as t increases in 10 regularly spaced intervals from $t = 0$ to $t = \frac{\pi}{2}$ with the first time of singularity occuring at $t \approx \frac{3\pi}{20}$.

that the evolution by isothermal gauge gives an explicit representation formula for a solution to the IVP and that singularities correspond to points where this formula fails to give an immersion. By analysing a Taylor expansion of the evolution by isothermal gauge around the point at which singularity first occurs, Eggers & Hoppe observed that for generic smooth initial data the evolution by isothermal gauge at the first time of singularity will look like a swallowtail (Figure 1.3). In the swallowtail, at the first time of singularity $t_* > 0$ the spatial cross section $\{x^0 = t_*\}$ is a $C^{1,1/3}$ curve, whilst for $t > t_*$ the spatial cross section $\{x^0 = t\}$ has a pair of ordinary cusps. In particular, in this generic case the evolution by isothermal gauge does not give a C^1 evolution beyond the first singular time. For a more detailed description of generic singularity formation in isothermal gauge, we refer to Nguyen & Tian [59, Section 3].

Remark 1.11 (The swallowtail in other areas of maths)**.** The swallowtail is well known from the branch of mathematics known as singularity theory or catastrophe theory, which studies the points at which a smooth map fails to give an immersion (see Whitney [72] or e.g. Arnol'd [3]). The swallowtail singularity also occurs as the shape of a wave front in geometric optics (Neu [60, Chap. 4]). For discussion of the links between the singularities of timelike maximal surfaces and the eikonal equation of geometric optics see Eggers, Hoppe, Hynek & Suramishvilli [24] or Neu [58] and for discussion of links with some self-similar singularity formations which arise in fluid dynamics see Eggers & Suramlishvili [25].

Remark 1.12 (Singularities in higher dimensions and codimensions)**.** It may be seen that singularity formation for (1+1)-dimensional timelike maximal surfaces in $\mathbb{R}^{1+2}$ also describes cases of singularity formations for timelike maximal surfaces in higher dimensions and codimensions, by trivially taking cross products and exploiting the finite speed of propagation for wave equations. But this is certainly not all the interesting behaviour! For some recent work on singularity formation in higher dimensions and codimensions, we refer to Bahouri, Marachli & Perelman [5], Wong [75], and Yan [76], and for some beautiful numerical simulations of singularity formation for codimension 1 timelike maximal surfaces in $\mathbb{R}^{1+3}$, we refer to Eggers, Hoppe, Hynek & Suramlishvili [24].

1.3.2 Stability results

In the last section we highlighted some literature on singularity formation for timelike maximal surfaces in $\mathbb{R}^{1+2}$ which is of particular relevance to our work. We will now proceed to briefly discuss some further results on timelike maximal surfaces, starting in this section with stability results in all dimensions and codimensions.

The global existence of solutions to the IVP for timelike maximal surfaces in

Minkowski space with initial data lying sufficiently close (in a Sobolev sense) to planar initial data together with the asymptotic convergence of solutions to said plane, was established by Brendle [16] (for $(1+d)$-dimensional codimension 1 timelike planes for $d \geq 3$), by Lindblad [50] (for $(1+d)$-dimensional codimension 1 timelike planes for $d \geq 1$ with asymptotic convergence only in the case $d \geq 2$), and by Allen, Andersson & Isenberg [1] (for $(1+d)$-dimensional timelike planes of arbitrary codimension for $d \geq 2$). Wong has given some improved results on the necessary decay required at infinity to prove global existence for small initial data for timelike maximal surfaces in $\mathbb{R}^{1+2}$ by energy methods [74]. With regards to an IBVP for timelike maximal surfaces in $\mathbb{R}^{1+2}$, the global existence of solutions to the Born-Infeld equation (1.6) for small Dirichlet initial-boundary data was established by Liu & Zhou on the quadrant $[0, \infty) \times [0, \infty)$ [51] as well as on the strip $[0, 1] \times [0, \infty)$ [52] with boundary data both small and decaying for the latter case. See also Sun [69] for some improved results on the two-boundary case including a Neumann boundary condition and a mixed Dirichlet-Neumann boundary condition (again with small data with respect to a weighted decay norm). Finally, let us mention the interesting work of Donninger, Krieger, Szeftel & Wong [22] who studied the stationary catenoidal solution to the codimension 1 timelike maximal surface equations in $\mathbb{R}^{1+3}$. Those authors identified an unstable mode of the linearized equation and proved the existence, in a neighbourhood of the catenoidal initial data in a certain symmetry class, of a codimension 1 Lipschitz manifold transverse to the unstable mode consisting of initial data for which global solutions exist and converge asymptotically to the catenoid.

Remark 1.13 (Comparison with the elliptic setting)**.** The stability of the timelike plane discussed above shows that there exist many smooth graphical properly embedded timelike maximal surfaces in $\mathbb{R}^{1+2}$. This is in contrast with the situations for minimal surfaces in $\mathbb{R}^3$ (i.e. Bernstein's theorem) and spacelike maximal surfaces in

$\mathbb{R}^{1+2}$ (i.e. Calabi [19]).

1.3.3 Further relevant results

With regards to weak formulations of the IVP, Brenier studied compactness properties of spatially non-compact timelike maximal surfaces and introduced the "subrelativistic string" as a generalized notion of solution [17]. This work was extended to the spatially compact setting by Belletini, Hoppe, Novaga & Orlandi in [8]. Timelike maximal surfaces have been studied as limits of concentration sets of the stationary points of a class of hyperbolic Ginzburg-Landau type functionals, see Neu [58], Belletini, Novaga & Orlandi [9], and Jerrard [41]. Belletini, Novaga & Orlandi have introduced notions of generalized timelike maximal surfaces using the language of varifolds and have shown, in particular, that these include the category of subrelativistic strings [10]. For work on the static Born-Infeld equation with a distribution of point charges, we refer to Bonheure, Colasuonno & Földes [12] and the references therein.

Whilst there exist no smooth proper timelike maximal immersions $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$, Nguyen & Tian gave an example of a smooth proper timelike maximal immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+3}$ and conjectured that generic closed curves evolve to globally regular surfaces in higher codimension [59, Appendix]. This conjecture was proved by Jerrard, Novaga & Orlandi in [42], where it was shown that for $n \geq 4$ the evolution of a generic spacelike closed curve in $\mathbb{R}^{1+n}$ in a generic timelike direction by isothermal gauge yields a proper timelike maximal immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+n}$ whilst in the borderline case $n = 3$ there are distinct, non-empty open sets of initial data leading to both regular surfaces and singular surfaces respectively.

For (1+1)-dimensional timelike maximal surfaces in curved spacetimes (important for physical theories incorporating gravity) it is still possible to solve the equations in isothermal gauge, see Gu [33]. This leads to an analysis of (1+1) wave maps into Lorentzian target manifolds. It may be seen to follow from the work of Gu [32] that the equations for a (1+1) wave map into a static Lorentzian target give globally regular solutions, and some further results on regularity of (1+1) wave maps into non-static targets were obtained by Müller in his PhD thesis, see [57].

For lots of interesting work on timelike maximal surfaces in all dimensions and codimensions and in particular for discussion on physical applications (i.e. quantized models) we refer to the survey article of Hoppe [38] as well as the many more papers of Hoppe [35]–[37] and of Hoppe and his collaborators [2, 8, 13, 14, 23, 39, 40], together with the references therein. To name just a few works from the body of work on spacelike maximal surfaces in Minkowski space, one may see Bartnik [7], Calabi [19], and Lambert [48] and the references therein. For interesting work on spacelike mean curvature flow in pseudo-Euclidean spaces of arbitrary signature, see Lambert & Lotay [49], and for maximal surfaces in Minkowski space of mixed type (i.e. with both timelike and spacelike parts) we refer to Gu [34] or to the more recent work of Fujimori et. al. [29] and the references therein. For recent work on the Born-Infeld equation and non-uniqueness of maximal globally hyperbolic developments, we refer to Eperon, Real & Spierski [28]. For the "hyperbolic mean curvature flow" and other related geometric PDE, see the work of Kong et. al. (e.g. [45] and the references therein). Finally, for an intriguing account of timelike CMC surfaces in Minkowski space, we refer to Wong [73].

1.4 Statement of the main results of this thesis

1.4.1 Embeddedness of timelike maximal surfaces

We now turn to the main results of this thesis. Recall that it was proved by Nguyen & Tian [59] (following work of Hoppe [36]) that there exists no smooth proper timelike immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ with vanishing mean curvature. Thus, by Lemma 1.1, every smooth properly immersed timelike maximal surface in $\mathbb{R}^{1+2}$ is an immersed $\mathbb{R}^2$. In other words, every global solution to the IVP for timelike maximal surfaces is a Cauchy evolution of a non-compact curve. Moreover, from the stability of the timelike plane (recall §1.3.2) we know that there exist many properly embedded graphical timelike maximal surfaces in $\mathbb{R}^{1+2}$ close to a timelike plane (i.e. many global solutions in the spatially non-compact case). Our first result concerns the geometry of spatially non-compact timelike maximal surfaces. We prove:

Theorem 2.3. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike immersion with vanishing mean curvature. Then ϕ is an embedding. Moreover, for each compact subset $K \subseteq \phi(\mathbb{R}^2)$ there is a timelike plane $P \subseteq \mathbb{R}^{1+2}$ such that K is a smooth graph over P.*

Let us gather a few remarks about Theorem 2.3.

Remark 1.14 (Sharpness of Theorem 2.3)**.** In §2.3 we will see examples of smooth proper timelike maximal embeddings $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ for which $\phi(\mathbb{R}^2)$ is a smooth graph as well as examples of smooth proper timelike maximal embeddings $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ for which $\phi(\mathbb{R}^2)$ is not a graph. The latter examples show that the restriction to compact subsets in Theorem 2.3 cannot be relaxed in general.

Remark 1.15 (The conformal structure of timelike maximal surfaces)**.** Crucial to the proof of Theorem 2.3 will be the construction of a smooth conformal equivalence between any smooth properly immersed timelike maximal surface and the Minkowski plane $\mathbb{R}^{1+1}$, see Lemma 2.2. The remarkable fact that such a conformal equivalence always exists was known already from the work of Milnor [56], although for our proof of Lemma 2.2 we will adapt an argument of Belletini, Hoppe, Novaga & Orlandi [8] who proved the corresponding statement for spatially compact timelike maximal surfaces. In contrast with the Riemannian setting, there are infinitely many conformal structures of simply-connected Lorentzian surfaces (Kulkarni [46]) so the existence of such a conformal equivalence for any properly immersed timelike maximal surface is far from obvious.

Remark 1.16 (The unit normal and a comparison with the Riemannian setting)**.** In terms of a spacelike unit normal along ϕ

$$N\colon \mathbb{R}^2 \to S^{1+1} = \left\{(\sinh\varphi, \cos\vartheta\cosh\varphi, \sin\vartheta\cosh\varphi) \in \mathbb{R}^{1+2}\colon (\vartheta,\varphi) \in [0,2\pi)\times\mathbb{R}\right\},$$

Theorem 2.3 says that for every compact subset $K \subseteq \mathbb{R}^2$ the set $N(K)$ is contained in an open hemi-hyperboloid

$$S^{1+1}_+ = \left\{(\sinh\varphi, \cos\vartheta\cosh\varphi, \sin\vartheta\cosh\varphi) \in \mathbb{R}^{1+2}\colon (\vartheta,\varphi) \in (\vartheta_0 - \frac{\pi}{2}, \vartheta_0 + \frac{\pi}{2})\times\mathbb{R}\right\}$$

for some $\vartheta_0 \in \mathbb{R}$, which is a hemi-sphere with respect to the Minkowski metric. The image of N will be a single point if $\mathrm{Im}(\phi)$ is a plane and we will give many examples where it is a subset of S^{1+1} of non-empty interior, including examples where the closure of $\mathrm{Im}(N)$ intersects both connected components of the boundary of a closed hemi-hyperboloid in S^{1+1} (see §2.3). This could be compared with the counterpart in the Riemannian setting, where we recall that for any complete minimal surface in $\mathbb{R}^3$ the image of the unit normal vector is either a single point or it omits at most 4 points in the sphere S^2 [30].

Remark 1.17 (Failure of Theorem 2.3 in higher codimension)**.** Theorem 2.3 does not extend to higher codimension. Indeed, recall (§1.3.3) that Nguyen & Tian gave an example of a smooth proper timelike maximal immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+3}$ in [59, Appendix] and it is easy to see how this example may be adapted to give a smooth proper timelike maximal immersion $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+3}$.

1.4.2 Formation of singularities

Having proved Theorem 2.3, we will turn to the IVP for timelike maximal surfaces (Definition 1.2). It may be shown that under mild conditions on the initial data (C, V) there will exist a local solution to the IVP (see Corollary 4.13 later). On the other hand, it may be seen to follow from Theorem 2.3 that if $\mathrm{Im}(U_0)$ contains a closed semi-circle where U_0 denotes the unit tangent along C (or equivalently, if $\mathrm{Im}(C)$ contains a compact subset which is not a smooth graph) then there exists no global solution to the IVP. Thus in this case a Cauchy evolution of (C, V) must become singular in finite time, either in the future or the past. A priori, there are two possible mechanisms by which a singularity may occur: (1) the maximal surface fails to remain timelike, or (2) the maximal surface fails to remain smooth. Our proof of Theorem 2.3, however, does not shed any light on the nature of the singularity. Our next theorem addresses this issue. We prove:

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set such that for some $(s_0, t_0) \in \mathbb{R}^2$ and some $\varepsilon > 0$ one has $\{s_0\} \times [t_0 - \varepsilon, t_0) \subseteq \Omega$ and $(s_0, t_0) \in \partial\Omega$. Let $\phi\colon \bar{\Omega} \to \mathbb{R}^{1+2}$ be a C^1 map of the form $\phi(s,t) = (t, \gamma(s,t))$ where γ satisfies $\langle \gamma_s(s_0, t), \gamma_t(s_0, t)\rangle = 0$ for $t \in [t_0 - \varepsilon, t_0)$ such that $\phi|_\Omega$ is a C^2 timelike immersion. Write h for the mean curvature scalar of ϕ and $k(\cdot, t)$ for the curvature of the (planar) curve $\gamma(\cdot, t)$. Suppose $|h(s,t)| \leq C$ for $(s,t) \in \Omega$ and suppose that $|\gamma_t(s_0, t_0)|^2 = 1$ (so that the spacelike unit normal N along ϕ blows-up in Euclidean norm, $\lim_{t \uparrow t_0} |N(s_0, t)| = \infty$, and if ϕ is an* *immersion then ϕ is null at (s_0, t_0)). Then*

$$\int_{t_0-\varepsilon}^{t_0} |k(s_0, t)| dt = \infty.$$

Corollary 3.2 (C^2 inextendibility)**.** *Let $\varepsilon > 0$, $(s_0, t_0) \in \mathbb{R}^2$ and $\phi\colon (s_0 - \varepsilon, s_0 + \varepsilon) \times (t_0 - \varepsilon, t_0] \to \mathbb{R}^{1+2}$ be a C^1 immersion of the form $\phi(s,t) = (t, \gamma(s,t))$*[2] *such that $\phi|_{(s_0-\varepsilon, s_0+\varepsilon)\times(t_0-\varepsilon,t_0)}$ is C^2 and timelike with bounded mean curvature $|h(s,t)| \leq C$ for $(s,t) \in (s_0 - \varepsilon, s_0 + \varepsilon) \times (t_0 - \varepsilon, t_0)$ and such that ϕ is null at the point (s_0, t_0) (i.e. $\mathrm{Im}(d\phi_{(s_0,t_0)})$ is a null plane in $\mathbb{R}^{1+2}$). Then ϕ is not C^2.*

It is readily checked that the hypotheses for Theorem 3.1 are met for the evolution by isothermal gauge (Definition 1.7) and thus Theorem 3.1 gives a description of singularity formation for timelike maximal surfaces.

Remark 1.18 (Comparison with Taylor expansion methods)**.** Recall that we discussed in §1.3.1 some work of previous authors on singularity formation for timelike maximal surfaces in $\mathbb{R}^{1+2}$ (recall that, generically, the phenomena is descibed in isothermal gauge by the swallowtail of Figure 1.3) so let us compare Theorem 3.1 with the methods of singularity analysis discussed in §1.3.1. Those methods rely on a Taylor expansion of the explicit representation formula (i.e. the evolution by isothermal gauge) about the point of singularity. In order to make this analysis one must assume that certain terms in the Taylor expansion are non-vanishing, and so these methods apply only to generic cases. But there exist non-generic cases in which these methods cannot be applied. Indeed, it is possible to cook up examples of smooth initial data (C, V) such that, computing the Taylor expansion of the evolution by isothermal gauge $\phi\colon \Lambda^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$ of (C, V) about the point of first singular time, one finds that all terms in the Taylor expansion vanish (see Examples 4.14 and 4.15

[2]note that any causal surface admits a local parameterisation of this form by the implicit function theorem

for such cases). Theorem 3.1 does not rely on a representation formula (it applies to the more general setting of surfaces of bounded mean curvature, where no representation formula is available) and it applies to the general case of singularity formation, rather than the generic case.

1.4.3 Evolution by isothermal gauge and C^1 inextendibility

From Theorem 2.3 it follows that if (C, V) is an initial data for which $\mathrm{Im}(U_0)$ contains a closed semi-circle where U_0 denotes the unit tangent vector along C, then there exists no global solution to the IVP. From Corollary 3.2 it then follows that a Cauchy evolution of (C, V) will be C^2 inextendible beyond some singular time. However, it is still possible that there may exist a C^1 extension beyond singular time. Indeed, recall (§1.3.1) that for generic smooth initial data, at the first time of singularity the limit curve will be of regularity $C^{1,1/3}$. A complete understanding of C^1 extendibility, independent of gauge, is currently out of our reach. Nonetheless, we will proceed in Chapter 4 to discuss one well-known extension beyond singular time: that given by solving the equations globally in time by isothermal gauge (this is the method of extension most popular in the physics literature, see e.g. [77, Chap. 7]).

Recall that for generic smooth initial data, the evolution by isothermal gauge in a neighbourhood of the first singularity will look like a swallowtail (Figure 1.3). So, generically, the evolution by isothermal gauge will not give a C^1 extension beyond singular time. However, it turns out that there do exist (non-generic) classes of initial data for which the evolution by isothermal gauge gives a global C^1 extension beyond singular time. We present two such examples in Examples 4.14 and 4.15. In Example 4.14, a class of smooth initial data (C, V) is given for which the evolution by isothermal gauge $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ of (C, V) is such that $\Sigma = \mathrm{Im}(\phi)$ is a C^1 embedded causal surface which is a smooth timelike maximal surface away from a pair of null lines. In this example Σ contains non-graphical compact subsets (compare with Theorem 2.3). In Example 4.15, a class of smooth initial data (C, V) is given for which the evolution by isothermal gauge $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ of (C, V) is such that $\Sigma = \mathrm{Im}(\phi)$ is a C^1 embedded doubly-periodic causal surface which is a smooth timelike maximal surface away from a rectangular lattice of null points. In this example Σ is a graph, but not a C^1 graph. See Figures 4.3 and 4.4 respectively for numerical approximations of Examples 4.14 and 4.15.

In both of Examples 4.14 and 4.15, the spatial unit tangent vector ${}^{\phi_s(s,t)}/_{|\phi_s(s,t)|}$ (defined a priori only on $\mathbb{R}^2 \setminus \mathcal{K}$) admits a continuous extension to a unit tangent vector field along ϕ, and a common feature of both examples is that $\mathrm{Im}(U_0)$ is exactly a closed semi-circle. It turns out that this behaviour is borderline. We prove:

Theorem 4.16. *Let (C, V) be a $C^1 \times C^0$ initial data where $C\colon \mathbb{R} \to \mathbb{R}^{1+2}$ is a proper immersion and let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$, $\phi(s,t) = (t, \gamma(s,t))$ be the evolution of (C, V) by isothermal gauge (see §4.1 for the definition in low regularity). Writing $U_0\colon \mathbb{R} \to S^1$ for the unit tangent vector along the initial curve $\gamma(\cdot, 0)$, suppose that $\mathrm{Im}(U_0)$ contains an arc of length $> \pi$. Then there exists an open interval $I \subseteq \mathbb{R}$ such that for every $t_* \in I$ either $\mathrm{Im}(\gamma(\cdot, t_*))$ is not a C^1 immersed curve, or $\mathrm{Im}(\gamma(\cdot, t_*))$ is a C^1 immersed curve but the spatial unit tangent $U(\cdot, t_*) = {}^{\gamma_s(\cdot,t_*)}/_{|\gamma_s(\cdot,t_*)|}$ (defined only on the set $\{s\colon \gamma_s(s, t_*) \neq 0\}$) admits no extension to a continuous unit tangent vector field along $\gamma(\cdot, t_*)$.*

A couple of remarks about Theorem 4.16:

Remark 1.19 (Comparison with work of previous authors)**.** In the case that the initial curve $C\colon S^1 \to \mathbb{R}^{1+2}$ is closed, the corresponding statement (i.e. the discontinuity of the spatial unit tangent) was proved by Nguyen & Tian [59, Prop. 2.9 & Prop. 2.11] for smooth initial data and by Jerrard, Novaga and Orlandi [42, Thm 5.1] for $C^1 \times C^0$ initial data. Theorem 4.16 extends the work of those authors to the spatially non-compact case. Note that if C is closed or self-intersecting then necessarily $\mathrm{Im}(U_0)$ contains an arc of length $> \pi$ (see Lemma 4.7 for a proof of this elementary fact). For the proof of Theorem 4.16 we rely on a technical lemma (Lemma 4.18) and for the proof of this lemma we follow [42].

Remark 1.20. In most cases, the discontinuity of the spatial unit tangent corresponds to the curve $\mathrm{Im}(\gamma(\cdot, t_*))$ failing to be C^1. There do exist, however, degenerate cases for which $\mathrm{Im}(\gamma(\cdot, t_*))$ is a C^1 curve but the unit tangent admits no continuous extension along $\gamma(\cdot, t_*)$, as we will show in Example 4.20. We have no example for which this situation occurs whilst $\mathrm{Im}(\phi)$ is a C^1 surface (globally) but we cannot rule this out.

In Chapter 4 we also present some more detailed analysis of the evolution by isothermal gauge. We observe that Theorem 3.1 may be applied directly in the context of the isothermal gauge and we combine this with a localized singularity statement (Proposition 4.5) to complement Theorem 2.3. We also observe that there exist examples of smooth initial data (C, V) where the curve C is self-intersecting for which the evolution by isothermal gauge is singular only in the past (Remark 4.8). We also present local and global existence results for timelike maximal surfaces which are notable in that they require no decay on the initial data at infinity (Corollary 4.13 and Remark 4.10).

1.4.4 Initial-boundary value problems

In Chapters 5 & 6 of this thesis, we consider initial-boundary value problems (IBVPs) for timelike maximal surfaces in $\mathbb{R}^{1+2}$. A notable example of a physical theory which considers IBVPs for timelike maximal surfaces is the bosonic string with endpoints attached to D-branes (the D stands for Dirichlet), see Johnson [43, Chap. 4].

In Chapter 5 we consider an IBVP for a timelike maximal surface with timelike boundary a future-directed timelike curve (which is an arbitrary timelike curve). We first fix a conformal structure of the solution and derive a C^2-compatibility condition which this imposes on the initial-boundary data. We then define a notion of the evolution by isothermal gauge for this IBVP and we prove that this gives a C^2 proper map which is a conformal timelike maximal immersion away from a (possibly empty) singular set (Proposition 5.5). We analyse the behaviour of the evolution by isothermal gauge in a neighbourhood of a singular point and show that there is always a curvature blow up (Lemma 5.8). It should be noted that Lemma 5.8 is suboptimal in comparison with our situation for the IVP where Theorem 3.1 may be applied directly. We present a simple example which illustrates both possible conformal structures of null infinity (Example 5.10) and we derive a sufficient condition on initial-boundary data for the singular set to be empty (Proposition 5.12). This condition implies in particular a C^1 stability result for the quadrant of a timelike plane (Remark 5.13) and the condition is non-perturbative, in the sense that there exist initial-boundary data which just fail to meet the condition for which the singular set is non-empty (Remark 5.14). We also treat in more detail the special case where the prescribed timelike boundary is a half-line, where a representation formula may be written explicitely. In particular, we give an example in this case (Example 5.17) where singularity forms only outside of the domain of dependence of the initial-curve (i.e. singularity forms as a result of waves “reflecting off the boundary”).

In Chapter 6 we briefly discuss applications to two further IBVPs: (i) for a timelike maximal surface which intersects a given timelike surface orthogonally along a single timelike boundary curve, and (ii) for a timelike maximal surface whose timelike boundary consists of a pair of prescribed timelike curves. For (i) (which may be viewed as a Neumann boundary condition), to keep the presentation simple, we treat only the case that the boundary surface is a timelike plane, although the method we develop applies to any timelike boundary surface. Interestingly, we show that the IBVP (i) may be reduced, by isothermal gauge considerations, to the one considered in Chapter 5 (in other words we construct a Neumann to Dirichlet map). For (ii) we treat only the case that the timelike boundary curves are parallel straight lines. In fact the author does not know how to treat cases other than this (with the exception of a pair of boundary curves obtained as small, weighted perturbations of a pair of timelike parallel lines) as the representation formulas become too complicated to analyse. In both of (i) and (ii) we can fully classify solutions via an explicit representation formula. Let us conclude this overview by stating in advance for the reader just the latter result concerning case (ii).

Theorem 6.11. *Let* $(C, V, \Gamma_1, \Gamma_2)$ *be an initial-boundary data where the timelike boundary curves* $\Gamma_1, \Gamma_2 \colon \mathbb{R} \to \mathbb{R}^{1+2}$ *are a pair of parallel straight lines* $\Gamma_1(x^0) = (x^0, 0, 0)$*,* $\Gamma_2(x^0) = (x^0, 1, 0)$ *and where the initial data* (C, V) *satsify appropriate* C^2 *compatibility conditions (see* (6.17) *later) and let* $\phi \colon [0, \lambda] \times \mathbb{R} \to \mathbb{R}^{1+2}$ *be the evolution of* $(C, V, \Gamma_1, \Gamma_2)$ *by isothermal gauge (see Definition 6.10 later). Write* $A_\pm(s) = (1, a_\pm(s))$ *for the future-directed null vector fields along* C *such that*

$$\operatorname{span}\Big\{A_+(s), A_-(s)\Big\} = \operatorname{span}\Big\{C'(s), V(s)\Big\}.$$

Then ϕ *is a* C^2 *immersion iff there exist a pair of disjoint open semi-circles* $\Lambda_+ \subseteq S^1$ *and* $\Lambda_- \subseteq S^1$ *such that* $\mathrm{Im}(a_+) \subseteq \Lambda_+$ *and* $\mathrm{Im}(a_-) \subseteq \Lambda_-$*. Moreover if* ϕ *is a* C^2 *immersion then* ϕ *is an embedding,* $\mathrm{Im}(\phi)$ *is a* C^2 *graph over some timelike plane, and* $\mathrm{Im}(\phi)$ *is invariant under the action on* $\mathbb{R}^{1+2}$ *by the group of isometries generated by the "corkscrew" motion* $Q(x^0, x^1, x^2) = (x^0 + \lambda, -x^1 + 1, -x^2)$ *(*Q *is a combination of a translation of* $\mathbb{R}^{1+2}$ *in time (i.e.* $(x^0, x^1, x^2) \mapsto (x^0 + \lambda, x^1, x^2)$*) and a spatial rotation of* $\mathbb{R}^{1+2}$ *by* π *radians leaving invariant the line* $\{(x^0, \frac{1}{2}, 0) \colon x^0 \in \mathbb{R}\}$ *(i.e.*

$(x^0, x^1, x^2) \mapsto (x^0, -x^1 + 1, -x^2)$) *and so* Q *satisfies, in particular, the translation identity* $Q^2(x^0, x^1, x^2) = (x^0 + 2\lambda, x^1, x^2)$)*.*

Chapter 2

Embeddedness of timelike maximal surfaces

2.1 Construction of global isothermal coordinates

In this section we will prove the following two lemmas:

Lemma 2.1. *Let* $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ *be a smooth proper timelike immersion. Then there exists a smooth diffeomorphism* $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\phi' = \phi \circ \psi$ *is of the form* $\phi'(s,t) = (t, \gamma(s,t))$ *where* $\gamma = (\gamma^1, \gamma^2)$ *satisfies* $|\gamma_s|^2 = 1$*.*

Lemma 2.2 (Existence of a smooth conformal equivalence with $\mathbb{R}^{1+1}$)**.** *Let* $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ *be a smooth proper timelike immersion with vanishing mean curvature. Then there exists a smooth diffeomorphism* $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\phi' = \phi \circ \psi$ *is of the form* $\phi'(s,t) = (t, \gamma(s,t))$ *where* $\gamma = (\gamma^1, \gamma^2)$ *satisfies*

$$\langle \gamma_s, \gamma_t \rangle = 0 \tag{2.1}$$

$$|\gamma_s|^2 + |\gamma_t|^2 = 1 \tag{2.2}$$

$$\gamma_{tt} - \gamma_{ss} = 0. \tag{2.3}$$

Proof of Lemma 2.1. The proof is a standard argument which relies on the fact that ϕ^0 is a Morse function. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike immersion. For each $t \in \mathrm{Im}(\phi^0)$ write

$$C_t = \{(y^1, y^2) \in \mathbb{R}^2 \colon \phi^0(y^1, y^2) = t\}.$$

Since ϕ is timelike ϕ^0 can have no critical points. Thus C_t is a smooth submanifold of $\mathbb{R}^2$ for all $t \in \mathrm{Im}(\phi^0)$ by the implicit function theorem.

Let $g = \phi^*\eta$ be the induced Lorentzian metric on $\mathbb{R}^2$ and let $X = \nabla_g \phi^0$, which is a smooth nowhere-vanishing vector field on $\mathbb{R}^2$. Note that $\phi(C_t) = \mathrm{Im}(\phi) \cap \{x^0 = t\}$ is spacelike so with respect to g the submanifolds C_t are spacelike and thus X is a timelike vector field orthogonal to the submanifolds C_t.

Define $T = \frac{1}{g(X,X)} X$ and consider the flow of T. Let $p \in \mathbb{R}^2$ and let $\xi_p\colon (a,b) \to \mathbb{R}^2$ be the smooth inextendible integral curve of T through p so $\frac{d\xi_p}{ds}(s) = T(\xi_p(s))$ and $\xi_p(0) = p$. Then $\frac{d}{ds}\left(\phi^0(\xi_p(s))\right) = (d\phi^0)_{\xi_p(s)}(T(\xi_p(s))) = 1$ and so

$$\phi^0(\xi_p(s)) = \phi^0(p) + s. \tag{2.4}$$

We claim that $b = \infty$ and $a = -\infty$. Indeed, suppose we had $b < \infty$. Since the curve ξ_p is timelike and by (2.4), then $\phi(\xi_p([0,b)))$ would lie in the intersection of the time slab $0 \le t \le b$ with the future-directed light cone with vertex at the point $\phi(p)$ i.e. those points $(x^0, x^1, x^2) \in \mathbb{R}^3$ such that

$$\begin{aligned} (x^1 - \phi^1(p))^2 + (x^2 - \phi^2(p))^2 &\le (x^0 - \phi^0(p))^2 \\ \phi^0(p) \le x^0 &\le \phi^0(p) + b \end{aligned}$$

which is a compact set. Since ϕ is a proper map it would follow that the curve $\xi_p([0,b))$

would lie in a compact set. As T is smooth it would then follow that ξ_p could be smoothly extended up to $s = b$, contradicting inextendibility of ξ_p. So $b = \infty$ and similarly $a = -\infty$.

From (2.4) it is seen that the flow $p \mapsto \xi_p(t)$ maps C_0 diffeomorphically onto C_t for each t, thus we have shown $\mathrm{Im}(\phi^0) = \mathbb{R}$ and we have a foliation of $\mathbb{R}^2$ given by smooth curves C_t for $t \in \mathbb{R}$. We claim that each C_t is connected. Indeed, for $p, q \in C_0$ let $\omega\colon [0,1] \to \mathbb{R}^2$ be a continuous path with $\omega(0) = p$ and $\omega(1) = q$. Define $\hat{\omega}(s) = \xi_{\omega(s)}(-\phi^0(\omega(s)))$. So $\hat{\omega}(s) \in C_0$ for all $s \in [0,1]$ by (2.4) and $\hat{\omega}$ is a continuous path with $\hat{\omega}(0) = p$ and $\hat{\omega}(1) = q$. Thus C_0 and hence each C_t is connected.

Let C_0 be parameterised as $C_0(s)$ for $s \in (-\infty, \infty)$ and define $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi(s,t) = \xi_{C_0(s)}(t).$$

By the group property of the flow, it is seen that ψ gives a bijection. Standard results on smooth dependence on initial conditions for ODE show that ψ gives a smooth map and since T is nowhere vanishing and orthogonal to C_0, we have $\det(d\psi)(s,0) \neq 0$ for all $s \in \mathbb{R}$ and so it follows $\det(d\psi)(s,t) \neq 0$ for all $(s,t) \in \mathbb{R}^2$ (see eg. [20, Chapter 1]). Thus ψ is a diffeomorphism and we have $\phi' = \psi \circ \phi$ satisfies $\phi'(s,t) = (t, \gamma(s,t))$. Finally, since ϕ is proper it follows that $|\gamma(s,t)|^2 \to \infty$ as $s \to \pm\infty$ for each t. Thus we may pass to an arclength parameterisation to ensure the condition $|\gamma_s(s,t)|^2 = 1$. $\square$

Proof of Lemma 2.2. For this proof we adapt the arguments given in Belletini, Hoppe, Novaga & Orlandi [8] and Nguyen & Tian [59], where the corresponding statement was proved in the spatially compact case.

By Lemma 2.1 we may assume that ϕ is of the form $\phi(s,t) = (t, \gamma(s,t))$ where $|\gamma_s|^2 = 1$. Since ϕ is timelike we have the bound $|\gamma_t|^2 < 1$. Now, let $s' = s'(s,t)$, $t' = t$ denote a smooth coordinate change, with $\frac{\partial s'}{\partial s} > 0$, and set $\gamma'(s',t') = \gamma(s,t)$. We will choose these new coordinates so that

$$\langle \gamma'_{s'}, \gamma'_{t'} \rangle = 0. \tag{2.5}$$

By the chain rule

$$\gamma'_{s'} = \left(\frac{\partial s'}{\partial s}\right)^{-1} \gamma_s \tag{2.6}$$

$$\gamma'_{t'} = -\left(\frac{\partial s'}{\partial s}\right)^{-1} \left(\frac{\partial s'}{\partial t}\right) \gamma_s + \gamma_t, \tag{2.7}$$

and substituting expressions (2.6) and (2.7) and observing $|\gamma_s|^2 = 1$ we see that (2.5) will be satisfied provided

$$\frac{\partial s'}{\partial t} - \langle \gamma_s, \gamma_t \rangle \frac{\partial s'}{\partial s} = 0. \tag{2.8}$$

Equation (2.8) is a linear transport equation and may be solved by the method of characteristics. The solution s' is constant along characteristic curves $(s(t), t)$ where the $s(t)$ are solutions to

$$\dot{s}(t) = -\langle \gamma_s(s(t), t), \gamma_t(s(t), t) \rangle. \tag{2.9}$$

Since the right hand side of (2.9) is smooth and since we have the a-priori bound

$$|\langle \gamma_s, \gamma_t \rangle| < 1, \tag{2.10}$$

smooth solutions to (2.9) exist for all $t \in \mathbb{R}$ and for each (s_0, t_0) there exists a unique characteristic through (s_0, t_0) which crosses through the line $\{t = 0\}$ precisely once. Thus for any smooth function $\rho \colon \mathbb{R} \to \mathbb{R}$ there is a unique smooth solution s' to (2.8) satisfying the Cauchy data

$$s'(s,0) = \rho(s).$$

The choice of Cauchy data ρ will be fixed later. For now, observe that that the condition $\frac{\partial s'}{\partial s} > 0$ is equivalent to

$$\dot{\rho}(s) > 0, \tag{2.11}$$

and by the uniform bound on the characteristic speed (2.10) we have $s'(s,t) \to \pm\infty$ as $s \to \pm\infty$ for all $t \in \mathbb{R}$ provided

$$\rho(s) \to \pm\infty \tag{2.12}$$

as $s \to \pm\infty$ holds. A smooth diffeomorphism $\psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ is thus well defined by $\psi^{-1}(s,t) = (s'(s,t),t)$ provided ρ is chosen so that (2.11) and (2.12) hold.

We have verified (2.5) (which is (2.1) in the (s',t') coordinates) and we proceed to show that a choice of ρ may be selected satisfying (2.11) and (2.12) so as to ensure (2.2) and (2.3). The timelike maximal surface equations (Appendix B) read

$$\partial_i(\sqrt{|\det(g)|}g^{i2}) = 0 \tag{2.13}$$

$$\partial_i(\sqrt{|\det(g)|}g^{ij}\partial_j\gamma) = 0. \tag{2.14}$$

Since the metric in the new coordinates is

$$g(s',t') = |\gamma'_{s'}|^2 ds'^2 + (-1 + |\gamma'_{t'}|^2)dt'^2$$

equation (2.13) reads

$$\partial_{t'}\sqrt{\frac{1-|\gamma'_{t'}|^2}{|\gamma'_{s'}|^2}}=0$$

which is equivalent to $1-|\gamma'_{t'}(s',t')|^2 = K(s')^2|\gamma'_{s'}(s',t')|^2$. Thus the condition

$$|\gamma'_{s'}(s',t')|^2+|\gamma_{t'}(s',t')|^2=1$$

will follow for all $(s',t')\in\mathbb{R}^2$ provided $\rho(s)$ is chosen such that

$$|\gamma'_{s'}(s',0)|^2+|\gamma'_{t'}(s',0)|^2=1$$

(i.e $K(s')^2=1$) for all $s'\in\mathbb{R}$. From (2.6), (2.7) and (2.8) we have

$$\begin{aligned}|\gamma'_{s'}(s',0)|^2+|\gamma'_{t'}(s',0)|^2 &= |\dot\rho(s)^{-1}\gamma_s(s,0)|^2+|\gamma_t(s,0)-\langle\gamma_s(s,0),\gamma_t(s,0)\rangle\gamma_s(s,0)|^2\\ &=\dot\rho(s)^{-2}+|\gamma_t(s,0)|^2-\langle\gamma_s(s,0),\gamma_t(s,0)\rangle^2\end{aligned}$$

which equals 1 provided

$$\dot\rho(s)=(1-|\gamma_t(s,0)|^2+\langle\gamma_s(s,0),\gamma_t(s,0)\rangle^2)^{-1/2}=|\det(g(s,0))|^{-1/2}.$$

Since ϕ is timelike, this ensures (2.11) and moreover by the bound

$$0<|\det(g(s,0))|\leq 1$$

we see

$$\rho(s)=\int_*^s(|\det(g(s,0))|)^{-1/2}\,ds\to\pm\infty$$

as $s \to \pm\infty$, which is (2.12). We have ensured (2.1) and (2.2) and as the metric now reads

$$g(s', t') = |\gamma'_{s'}(s', t')|^2 \left(ds'^2 - dt'^2\right),$$

the equation $\gamma'_{t't'} - \gamma'_{s's'} = 0$ follows from (2.14). This completes the proof. $\square$

2.2 Embeddedness of timelike maximal surfaces

In this section we will prove

Theorem 2.3. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike immersion with vanishing mean curvature. Then ϕ is an embedding. Moreover, for each compact subset $K \subseteq \phi(\mathbb{R}^2)$ there is a timelike plane $P \subseteq \mathbb{R}^{1+2}$ such that K is a smooth graph over P.*

To prove Theorem 2.3, in light of Lemma 2.2 let us consider a smooth proper timelike immersion $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ of the form

$$\phi(s, t) = (t, \gamma(s, t)) \tag{2.15}$$

where $\gamma = (\gamma^1, \gamma^2)$ satisfies

$$\langle \gamma_s, \gamma_t \rangle = 0 \tag{2.16}$$

$$|\gamma_s|^2 + |\gamma_t|^2 = 1 \tag{2.17}$$

$$\gamma_{tt} - \gamma_{ss} = 0. \tag{2.18}$$

Define

$$a_{\pm}(s) = \gamma_t(s,0) \pm \gamma_s(s,0), \tag{2.19}$$

so that $|a_{\pm}(s)|^2 = 1$ by (2.16)–(2.17) (i.e. $a_{\pm}$ give the spatial directions of the outgoing and incoming null tangent vectors to $\phi(\mathbb{R}^2)$ along the initial curve $\phi(\cdot, 0)$). The following Lemma shows that the images of the outgoing and incoming null directions must be disjoint on a smooth timelike properly immersed maximal surface.

Lemma 2.4. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth, proper, timelike immersion of the form* (2.15) *where γ satisfies* (2.16)–(2.18) *and define $a_{\pm}$ by* (2.19). *Then $a_+(\xi) \neq a_-(\eta)$ for all $\xi, \eta \in \mathbb{R}$.*

Proof. Since γ satisfies the wave equation (2.18) we have d'Alembert's formula

$$\gamma(s,t) = \frac{1}{2}\left(\gamma(s+t,0) + \gamma(s-t,0) + \int_{s-t}^{s+t} \gamma_t(\xi,0)\, d\xi\right). \tag{2.20}$$

Differentiating gives

$$\begin{aligned} \gamma_s(s,t) &= \frac{1}{2}\left(\gamma_s(s+t,0) + \gamma_s(s-t,0) + \gamma_t(s+t,0) - \gamma_t(s-t,0)\right) \\ &= \frac{1}{2}\left(a_+(s+t) - a_-(s-t)\right). \end{aligned} \tag{2.21}$$

Since ϕ is an immersion $\gamma_s(s,t) \neq 0$ for all $(s,t) \in \mathbb{R}^2$ and thus $a_+(\xi) \neq a_-(\eta)$ for all $\xi, \eta \in \mathbb{R}$ as claimed. $\square$

Remark 2.5. It may be observed that Lemma 2.4 is a direct consequence of the conformal structure of a properly immersed timelike maximal surface (Lemma 2.2) together with Property 1.5.

Lemma 2.6. *Let $M > 0$ and let $a_{\pm}\colon [-M, M] \to \mathbb{R}^2$ be continuous functions satisfying $|a_{\pm}|^2 = 1$ and $a_+(\xi) \neq a_-(\eta)$ for all $\xi, \eta \in [-M, M]$. Then there exists $\omega \in \mathbb{R}^2$,*

$|\omega|^2 = 1$ *such that*

$$\langle a_+(\xi) - a_-(\eta), \omega \rangle > 0 \tag{2.22}$$

for all $\xi, \eta \in [-M, M]$.

Proof. $A = \mathrm{Im}(a_+)$ is a non-empty connected closed proper subset of S^1, so we may write

$$A = \{(\cos\alpha, \sin\alpha) \colon \alpha \in [\alpha_1, \alpha_2]\}$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfying $0 < \alpha_2 - \alpha_1 < 2\pi$. Defining $\omega = (\cos\frac{\alpha_1+\alpha_2}{2}, \sin\frac{\alpha_1+\alpha_2}{2})$ it follows by trigonometry that $\langle a, \omega \rangle > \langle b, \omega \rangle$ for all $a \in A$, $b \in S^1 \setminus A$. Since it is assumed $\mathrm{Im}(a_-) \subseteq S^1 \setminus A$ the claim is proved. □

We now have the tools to hand to prove Theorem 2.3.

Proof of Theorem 2.3. Let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike immersion with vanishing mean curvature. By Lemma 2.2, we may take ϕ to be of the form $\phi(s,t) = (t, \gamma(s,t))$ where γ satisfies (2.16)–(2.18).

Let $M > 0$ and define the characteristic diamond

$$D_M = \{(s,t) \colon |s| + |t| \leq M\} \subseteq \mathbb{R}^2. \tag{2.23}$$

To prove the theorem, we will show that $\phi|_{D_M}$ is injective and $\phi(D_M)$ is a smooth graph over a timelike plane P_M. Since M is arbitrary, from this it will follow that ϕ is injective and thus an embedding. Since ϕ is proper, given any compact subset $K \subseteq \phi(\mathbb{R}^2)$, we may choose M sufficiently large such that $K \subseteq \phi(D_M)$, so that K will thus be a smooth graph over the plane P_M. The theorem will thus be proved.

Defining $a_\pm$ as in (2.19) and by Lemma 2.4 we have that $a_+(\xi) \neq a_-(\eta)$ for all $\xi, \eta \in \mathbb{R}$. So by Lemma 2.6 there exists $\omega_M \in \mathbb{R}^2$, $|\omega_M|^2 = 1$ such that

$$\langle a_+(\xi) - a_-(\eta), \omega_M \rangle > 0$$

for all $\xi, \eta \in [-M, M]$. From (2.21) it follows

$$\langle \gamma_s(s,t), \omega_M \rangle = \frac{1}{2} \langle a_+(s+t) - a_-(s-t), \omega_M \rangle > 0 \tag{2.24}$$

for all $(s,t) \in D_M$.

From (2.24) it is now routine to show that $\phi|_{D_M}$ is an embedding and there is a timelike plane $P_M \subseteq \mathbb{R}^{1+2}$ such that $\phi(D_M)$ is a smooth graph over P_M, but let us go through the argument for completeness. By choosing inertial coordinates (x^0, x^1, x^2) on $\mathbb{R}^{1+2}$ appropriately, we may assume for convenience that $\omega_M = (1,0)$. Then, in the new coordinates, keeping the same notation for the parameterisation, (2.24) reads

$$\gamma_s^1(s,t) > 0 \tag{2.25}$$

for all $(s,t) \in D_M$. Let P_M be the x^0–x^1 plane in these new coordinates.

Write $D'_M = \{(t, \gamma^1(s,t)) \colon (s,t) \in D_M\} \subseteq \mathbb{R}^2$ and let $F \colon D_M \to D'_M$ be defined by $F(s,t) = (t, \gamma^1(s,t))$. From (2.25) it follows by monotonicity that F is bijective, and moreover by the inverse function theorem that F is a smooth diffeomorphism. Inverting F as $F^{-1}(x^0, x^1) = (s(x^0,x^1), t(x^0,x^1))$ gives

$$\begin{aligned} \phi(D_M) &= \phi \circ F^{-1}(D'_M) \\ &= \Big\{ \Big(x^0, x^1, \gamma^2(s(x^0,x^1), t(x^0,x^1)) \Big) : (x^0,x^1) \in D'_M \Big\} \end{aligned} \tag{2.26}$$

so we have shown $\phi(D_M)$ is a smooth graph over the x^0–x^1 plane. Moreover, it follows from (2.26) that $\phi\circ F^{-1}\colon D'_M \to \mathbb{R}^{1+2}$ is injective, so $\phi|_{D_M}$ is injective. This completes the proof. $\square$

2.3 Examples of graphical and non-graphical smooth properly embedded timelike maximal surfaces

Example 2.7 (Smooth properly embedded graphical timelike maximal surfaces)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be any smooth function and let $G = \{(u, f(u))\colon u \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be the graph of f. Let $c\colon \mathbb{R} \to \mathbb{R}^2$ be a smooth parameterisation of G by arclength, so $\mathrm{Im}(c) = G$ and $|c'(s)| = 1$. Define $\gamma(s,t) = \frac{1}{2}\left(c(s+t) + c(s-t)\right)$ and $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ by $\phi(s,t) = (t, \gamma(s,t))$. It may be checked that ϕ defines a smooth proper timelike maximal embedding[1] and $\phi(\mathbb{R}^2)$ is a smooth graph over the x^0–x^1 plane with $\phi(\mathbb{R}^2) \cap \{x^0 = 0\} = G$.

Example 2.8 (Smooth properly embedded doubly-periodic graphical timelike maximal surfaces)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(0) = 0$ and $f(u) = f(u+1)$ for all $u \in \mathbb{R}$ (i.e. f is periodic with period 1). As in Example 2.7, let $c\colon \mathbb{R} \to \mathbb{R}^2$ parametrize the graph of f by arclength, and define $\gamma(s,t) = \frac{1}{2}\left(c(s+t) + c(s-t)\right)$ and $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ by $\phi(s,t) = (t, \gamma(s,t))$. Note that $c(s+L) = c(s) + (1,0)$, where L is the length of one period of f and necessarily $L \geq 1$ with equality if and only if $f \equiv 0$ (i.e. if and only if the graph of f is a straight line). Now let us observe that $\mathrm{Im}(\phi)$ is doubly periosic. Indeed, observe that $\phi(s+L, t) = \phi(s,t) + (0,1,0)$ and $\phi(s, t+L) = \phi(s,t) + (L,0,0)$. Thus, defining $T\colon \mathbb{R}^{1+2} \to \mathbb{R}^{1+2}$ by $T(x^0, x^1, x^2) = (x^0 + L, x^1, x^2)$ for a translation in time

[1] Note that ϕ here is precisely the evolution of (C, V) by isothermal gauge with initial velocity $V = (1,0,0)$

and $S\colon \mathbb{R}^{1+2} \to \mathbb{R}^{1+2}$ by $S(x^0, x^1, x^2) = (x^0, x^1 + 1, x^2)$ for a translation in space, we see $\phi(\mathbb{R}^2)$ is invariant under both T and S. Thus $\phi(\mathbb{R}^2)$ is periodic in the direction $(1, 0, 0)$ with period L and periodic in the direction $(0, 1, 0)$ with period 1.

Example 2.9 (Smooth properly embedded non-graphical timelike maximal surfaces.)**.** Let $c\colon \mathbb{R} \to \mathbb{R}^2$ be a parametrisation of a smooth curve by arclength such that the following hold:

1. $c(s) = (0, -s)$, for $s \in (-\infty, -1]$,
2. $c'^1(s) > 0$ for $s \in (-1, \infty)$,
3. as $s \to \infty$, $c'(s) \to (0, 1)$.

See Figure 2.1 for a rough illustration of such a curve. Note that every compact subset of $\mathrm{Im}(c)$ is a smooth graph but $\mathrm{Im}(c)$ is not a smooth graph. Define $\gamma(s, t) =$

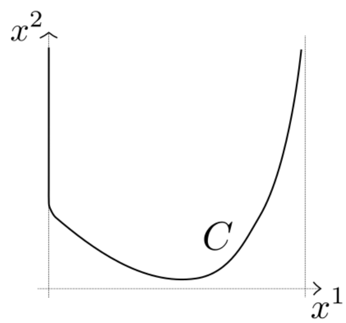


Figure 2.1: A smooth planar curve which is not a graph for which every compact subset is a smooth graph.

$\frac{1}{2}\left(c(s + t) + c(s - t)\right)$ and $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ by $\phi(s, t) = (t, \gamma(s, t))$. Then ϕ is a smooth proper timelike maximal embedding. For every compact subset $K \subseteq \phi(\mathbb{R}^2)$ there is a timelike plane $P \subseteq \mathbb{R}^{1+2}$ such that K is a smooth graph over P, which is consistent with Theorem 2.3, but we now claim that $\Sigma = \phi(\mathbb{R}^2)$ is not a graph. To see this, observe that $\phi(s, t) = (t, 0, -s)$ for $s \leq -1 - |t|$, so Σ contains an open quadrant $Q = \{(t, 0, -s)\colon s < -1 - |t|\}$ of the plane $\{x^1 = 0\}$. For each $t \in \mathbb{R}$, the curve

$s \mapsto \phi(s,t)$ asymptotes to the plane $\{x^1 = 1\}$ as $s \to \infty$, and Σ forms the boundary of an open subset of $\mathbb{R}^{1+2}$ which is sandwiched between the planes $\{x^1 = 0\}$ and $\{x^1 = 1\}$ (i.e. $\Sigma = \partial B$ where $B \subseteq \{0 < x^1 < 1\}$). It then follows that for every point $q \in Q \subseteq \Sigma$, every straight line in $\mathbb{R}^{1+2}$ through q intersects Σ at at least 2 distinct points. Thus Σ is not a graph. In this example, the image of the unit normal N is not contained in any open hemi-hyperboloid, but is contained in the union of an open hemi-hyperboloid with one connected component of its boundary.

Chapter 3

Curvature blow-up

3.1 C^2 inextendibility

We now turn to the question of whether it is possible to relax the notion of a timelike maximal surface, either by allowing for surfaces which are C^k for some $k \geq 1$ or by allowing for null points (i.e. degenerate hyperbolicity), in such a way as to continue beyond singular time in a Cauchy evolution. Our first result in this direction will be that if the evolution fails to remain timelike, then the maximal surface must fail to be C^2 immersed. In fact, we will deduce this from an observation which holds for more general evolutions of surfaces of only bounded mean curvature.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set such that for some $(s_0, t_0) \in \mathbb{R}^2$ and some $\varepsilon > 0$ one has $\{s_0\} \times [t_0 - \varepsilon, t_0) \subseteq \Omega$ and $(s_0, t_0) \in \partial\Omega$. Let $\phi \colon \bar{\Omega} \to \mathbb{R}^{1+2}$ be a C^1 map of the form $\phi(s,t) = (t, \gamma(s,t))$ where γ satisfies $\langle \gamma_s(s_0, t), \gamma_t(s_0, t) \rangle = 0$ for $t \in [t_0 - \varepsilon, t_0)$ such that $\phi|_\Omega$ is a C^2 timelike immersion. Write h for the mean curvature scalar of ϕ and $k(\cdot, t)$ for the curvature of the (planar) curve $\gamma(\cdot, t)$. Suppose $|h(s,t)| \leq C$ for $(s,t) \in \Omega$ and suppose that $|\gamma_t(s_0, t_0)|^2 = 1$ (so that the spacelike unit normal N along ϕ blows-up in Euclidean norm, $\lim_{t \uparrow t_0} |N(s_0, t)| = \infty$, and if ϕ is an* *immersion then ϕ is null at (s_0, t_0)). Then*

$$\int_{t_0-\varepsilon}^{t_0} |k(s_0,t)|dt = \infty. \tag{3.1}$$

Proof. By taking $\varepsilon > 0$ sufficiently small, we may ensure that $|\gamma_t(s,t)|^2 > 0$ for $(s,t) \in \Omega \cap B_\varepsilon(s_0,t_0)$. It may then be seen that a spacelike unit normal vector N to $\phi(\Omega \cap B_\varepsilon(s_0,t_0))$ is given along $\{s_0\} \times [t_0-\varepsilon, t_0)$ by

$$N(s_0,t) = \frac{1}{(1-|\gamma_t(s_0,t)|^2)^{1/2}} \begin{pmatrix} |\gamma_t(s_0,t)| \\ n(s_0,t) \end{pmatrix},$$

where

$$n(s_0,t) = {}^{\gamma_t(s_0,t)}\!/_{|\gamma_t(s_0,t)|}$$

is a unit normal to the planar curve $\gamma(\cdot,t)$ at the point $s = s_0$.

The curvature of the cross sections $\gamma(\cdot,t)$ is given at $s = s_0$ by

$$k(s_0,t) = \frac{\langle \gamma_{ss}(s_0,t), n(s_0,t)\rangle}{|\gamma_s(s_0,t)|^2}.$$

Along $\{s_0\} \times [t_0-\varepsilon, t_0)$, the components of the first fundamental form $E(s,t)ds^2 + 2F(s,t)dsdt + G(s,t)dt^2$ are calculated as

$$\begin{aligned} E(s_0,t) &= |\gamma_s(s_0,t)|^2 \\ F(s_0,t) &= \langle \gamma_s(s_0,t), \gamma_t(s_0,t)\rangle = 0 \\ G(s_0,t) &= -1 + |\gamma_t(s_0,t)|^2, \end{aligned}$$

the components of the second fundamental form $e(s,t)ds^2 + 2f(s,t)dsdt + g(s,t)dt^2$

are

$$\begin{aligned}
e(s_0,t) &= -\frac{\langle \gamma_{ss}(s_0,t), n(s_0,t)\rangle}{(1-|\gamma_t(s_0,t)|^2)^{1/2}} \\
f(s_0,t) &= -\frac{\langle \gamma_{st}(s_0,t), n(s_0,t)\rangle}{(1-|\gamma_t(s_0,t)|^2)^{1/2}} \\
g(s_0,t) &= -\frac{\langle \gamma_{tt}(s_0,t), n(s_0,t)\rangle}{(1-|\gamma_t(s_0,t)|^2)^{1/2}},
\end{aligned}$$

and the mean curvature scalar is

$$\begin{aligned}
h(s_0,t) &= \frac{e(s_0,t)}{E(s_0,t)} + \frac{g(s_0,t)}{G(s_0,t)} \\
&= -\frac{\langle \gamma_{ss}(s_0,t), n(s_0,t)\rangle}{|\gamma_s(s_0,t)|^2(1-|\gamma_t(s_0,t)|^2)^{1/2}} + \frac{\langle \gamma_{tt}(s_0,t), n(s_0,t)\rangle}{(1-|\gamma_t(s_0,t)|^2)^{3/2}}.
\end{aligned} \tag{3.2}$$

Rearranging (3.2) gives the identity

$$(1-|\gamma_t(s_0,t)|^2)^{1/2}h(s_0,t) + k(s_0,t) = \frac{\langle \gamma_{tt}(s_0,t), n(s_0,t)\rangle}{1-|\gamma_t(s_0,t)|^2}. \tag{3.3}$$

Next we claim that

$$\int_{t_0-\varepsilon}^{t_0} \frac{\langle \gamma_{tt}(s_0,t), n(s_0,t)\rangle}{1-|\gamma_t(s_0,t)|^2} = \infty. \tag{3.4}$$

To show (3.4), write $\mu(t) := |\gamma_t(s_0,t)|^2$ so that

$$\frac{\langle \gamma_{tt}(s_0,t), n(s_0,t)\rangle}{1-|\gamma_t(s_0,t)|^2} = \frac{\langle \gamma_{tt}(s_0,t), \gamma_t(s_0,t)\rangle}{|\gamma_t(s_0,t)|(1-|\gamma_t(s_0,t)|^2)} = \frac{\frac{1}{2}\dot{\mu}(t)}{\mu(t)^{1/2}(1-\mu(t))}.$$

We have by assumption $\mu(t)\uparrow 1$ as $t\uparrow t_0$, so

$$\int_{t_0-\varepsilon}^{t_0} \frac{\dot{\mu}(t)}{1-\mu(t)}dt = \int_{t_0-\varepsilon}^{t_0} -\frac{d}{dt}(\log(1-\mu(t)))dt = \infty$$

from which (3.4) follows. As $|h(s,t)|\leq C$, (3.1) then follows from (3.3) and (3.4) and the theorem is proved. $\square$

Corollary 3.2 (C^2 inextendibility)**.** *Let $\varepsilon > 0$, $(s_0, t_0) \in \mathbb{R}^2$ and $\phi\colon (s_0-\varepsilon, s_0+\varepsilon)\times(t_0-\varepsilon, t_0] \to \mathbb{R}^{1+2}$ be a C^1 immersion of the form $\phi(s,t) = (t, \gamma(s,t))$*[1] *such that $\phi|_{(s_0-\varepsilon,s_0+\varepsilon)\times(t_0-\varepsilon,t_0)}$ is C^2 and timelike with bounded mean curvature $|h(s,t)| \leq C$ for $(s,t) \in (s_0-\varepsilon, s_0+\varepsilon)\times(t_0-\varepsilon, t_0)$ and such that ϕ is null at the point (s_0,t_0) (i.e. $\mathrm{Im}(d\phi_{(s_0,t_0)})$ is a null plane in $\mathbb{R}^{1+2}$). Then ϕ is not C^2.*

Proof of Corollary 3.2. Let $\phi\colon (s_0-\varepsilon, s_0+\varepsilon)\times(t_0-\varepsilon, t_0] \to \mathbb{R}^{1+2}$, $\phi(s,t) = (t,\gamma(s,t))$ be a C^1 immersion which is a C^2 timelike immersion with bounded mean curvature on $(s_0-\varepsilon, s_0+\varepsilon)\times(t_0-\varepsilon, t_0)$ and which is null at the point (s_0,t_0). For a sufficiently small $\varepsilon_0 \in (0,\varepsilon)$, let $r\colon [t_0-\varepsilon_0, t_0] \to \mathbb{R}$ be a solution to the terminal value problem

$$\dot{r}(t) = -\frac{\langle \gamma_s(r(t),t), \gamma_t(r(t),t)\rangle}{|\gamma_s(r(t),t)|^2}; \quad r(t_0) = s_0,$$

which satisfies $|r(t) - s_0| < \frac{\varepsilon}{2}$ for all $t \in [t_0-\varepsilon_0, t_0]$ (such a solution exists by the Peano existence theorem, see e.g. [20, Chap. 1]). We have $r \in C^2([t_0-\varepsilon_0, t_0)) \cap C^1([t_0-\varepsilon_0, t_0])$.

Define $\Omega = (-\varepsilon_0, \varepsilon_0)\times(t_0-\varepsilon_0, t_0)$. Let $\phi'\colon \bar{\Omega} \to \mathbb{R}^{1+2}$, $\phi'(s',t') = (t', \gamma'(s',t'))$ where $\gamma' = (\gamma'^1, \gamma'^2)$ is given by $\gamma'(s',t') = \gamma(r(t')+s', t')$. Then ϕ' is a C^1 immersion which is a C^2 timelike immersion with bounded mean curvature on Ω and $\phi'(0,t_0) = \phi(s_0,t_0)$. By the chain rule,

$$\langle \gamma'_{s'}(s',t'), \gamma'_{t'}(s',t')\rangle = \dot{r}(t')|\gamma_s(r(t')+s',t')|^2 + \langle \gamma_s(r(t')+s',t'), \gamma_t(r(t')+s',t')\rangle$$

so by construction we have $\langle \gamma'_{s'}(0,t'), \gamma'_{t'}(0,t')\rangle = 0$ for $t' \in (t_0-\varepsilon_0, t_0)$. As ϕ' is null at $(0,t_0)$, it may be seen that $|\gamma'_{t'}(0,t_0)|^2 = 1$. So since $|h(s',t')| \leq C$ for $(s',t') \in \Omega$ we see ϕ' satisfies the conditions for Theorem 3.1, so $\limsup_{t'\uparrow t_0} |k(0,t')| = \infty$ where

[1] note that any causal surface admits a local parameterisation of this form by the implicit function theorem

$k(\cdot,t')$ is the curvature of the planar cross sections $\gamma'(\cdot,t')$. Thus the curvatures of the curves $\gamma(\cdot,t)$ are not uniformly bounded for $t\in[t_0-\varepsilon,t_0]$ so ϕ is not C^2. $\square$

3.2 The shrinking circle revisited

Example 3.3 (Curvature blow-up for the shrinking circle)**.** Let us revisit the shrinking circle of Example 1.10. Define $\phi\colon S^1\times(-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}^{1+2}$ by $\phi(s,t)=(t,\gamma(s,t))$ where $\gamma(s,t)=(\cos t\cos s,\cos t\sin s)$. Then $h(s,t)=0$ and ϕ is a timelike maximal immersion. In addition, $\langle\gamma_s,\gamma_t\rangle=0$ (the parameterisation is orthogonal) and $|\gamma_t(s,t)|^2\uparrow 1$ as $t\uparrow\frac{\pi}{2}$. Observe $|k(s,t)|=|\cos t|^{-1}$, and $\int_0^{\frac{\pi}{2}}|k(s,t)|dt=\infty$ for all s, which is consistent with (3.1). For this example, we may study the rate of curvature blow-up in more detail. The element of arclength along $\gamma(\cdot,t)$ is $dl(s)=|\cos t|ds$, so for $p,q\in(1,\infty)$ one has

$$\begin{aligned}\|k\|_{L^q((0,\frac{\pi}{2});L^p(S^1))}&=\left(\int_0^{\frac{\pi}{2}}\left(\int_0^{2\pi}|k(s,t)|^p dl(s)\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}}\\&=(2\pi)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{2}}|\cos t|^{\frac{q(1-p)}{p}}dt\right)^{\frac{1}{q}}\end{aligned}$$

and since $\frac{q(1-p)}{p}\le -1$ iff $\frac{1}{p}+\frac{1}{q}\le 1$ we deduce that

$$\|k\|_{L^q((0,\frac{\pi}{2});L^p(S^1))}=\infty \quad \text{iff} \quad \frac{1}{p}+\frac{1}{q}\le 1.$$

Remark 3.4 (Inextendibility of the shrinking circle)**.** The shrinking circle of Example 3.3 is future (resp. past) C^1 inextendible beyond the singular time $\frac{\pi}{2}$ (resp. $-\frac{\pi}{2}$). In fact, the maximal extension of $\phi\left((S^1\times(-\frac{\pi}{2},\frac{\pi}{2})\right)$ to a C^0 submanifold of $\mathbb{R}^{1+2}$ is given by taking the closure of $\phi\left(S^1\times(-\frac{\pi}{2},\frac{\pi}{2})\right)$ in $\mathbb{R}^{1+2}$ i.e. by attaching one point at $x^0=\frac{\pi}{2}$ and one point at $x^0=-\frac{\pi}{2}$. In §4.3 we will see examples where the evolution is C^2 inextendible but C^1 extendible.

Remark 3.5 (Conformal structure of the shrinking circle)**.** It is an interesting exercise to observe that the shrinking circle $\phi\left(S^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})\right) \subseteq \mathbb{R}^{1+2}$ of Example 3.3 is smoothly conformally equivalent to the de Sitter sphere $S^{1+1} = \{-(x^0)^2 + (x^1)^2 + (x^2)^2 = 1\} \subseteq \mathbb{R}^{1+2}$.

Chapter 4

C^1 extendibility

4.1 Evolution by isothermal gauge

We will now proceed to concern ourselves with C^1 evolutions by isothermal gauge, so let us briefly review the method of isothermal gauge (§1.2.3) for $C^1 \times C^0$ initial data.

Let $C\colon \mathbb{R} \to \mathbb{R}^{1+2}$ be a C^k ($k \geq 1$) proper immersion of the form $C(s) = (0, c(s))$ and let V be a C^{k-1} future-directed timelike vector field along C. We call (C, V) an initial data. From the point of view of the IVP the prescription of initial data is equivalent to a prescription of a C^k curve C and a C^{k-1} distribution of timelike tangent planes along C, so without loss of generality we may assume V is of the form $V(s) = (1, v(s))$ where $\langle c'(s), v(s)\rangle = 0$. Since V is timelike implies $|v(s)| < 1$ we may then reparametrize the curve $C(s)$ to ensure the additional constraint $|c'(s)|^2 + |v(s)|^2 = 1$ holds. The pair $(C'(s), V(s))$ now gives an orthogonal frame along the initial data and the timelike planes $\operatorname{span}\{C'(s), V(s)\}$ are spanned by the future-directed null vectors

$$A_{\pm}(s) = V(s) \pm C'(s) = (1, v(s) \pm c'(s)) = (1, a_{\pm}(s)).$$

We say that the initial data is parametrized isothermally. Define a C^k map $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ by $\phi(s,t) = (t, \gamma(s,t))$ where $\gamma = (\gamma^1, \gamma^2)$ is given by d'Alembert's formula

$$\gamma(s,t) = \frac{1}{2}\left(c(s+t) + c(s-t) + \int_{s-t}^{s+t} v(\zeta)d\zeta \right). \tag{4.1}$$

The formula (4.1) implies that $\gamma_{tt} - \gamma_{ss} = 0$, $\gamma(s,0) = c(s)$, $\gamma_t(s,0) = v(s)$ with the wave equation understood in the weak sense when γ is not C^2. The isothermal gauge conditions $|\gamma_t(s,t) \pm \gamma_s(s,t)|^2 = 1$ are satisfied for all $(s,t) \in \mathbb{R}^2$ by (4.1) and we call $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ the evolution of (C,V) by isothermal gauge.

Write $\Sigma = \phi(\mathbb{R}^2)$ and define the closed (possibly empty) singular set by

$$\mathcal{K} = \{(s,t) \in \mathbb{R}^2 \colon \gamma_s(s,t) = 0\} = \Big\{(s,t) \in \mathbb{R}^2 \colon a_+(s+t) = a_-(s-t)\Big\} \tag{4.2}$$

so that ϕ gives a C^k immersion on $\mathbb{R}^2 \setminus \mathcal{K}$. Moreover, it may be seen that ϕ defines a C^k timelike maximal immersion on $\mathbb{R}^2 \setminus \mathcal{K}$ which is a conformal map with respect to the metric $ds^2 - dt^2$ on $\mathbb{R}^2 \setminus \mathcal{K}$. We write

$$\Sigma_{\text{sing}} = \phi(\mathcal{K}).$$

By continuity $\Sigma \setminus \Sigma_{\text{sing}}$ contains a neighbourhood of $\mathrm{Im}(C)$ and by construction $\Sigma \setminus \Sigma_{\text{sing}}$ gives a C^k timelike maximal immersed surface containing C and tangent to the vector field V along C. The following simple topological result shows that this is indeed a global evolution.

Lemma 4.1. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$, $\phi(s,t) = (t, \gamma(s,t))$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C,V) where $C = \gamma(\cdot, 0)$ is a proper immersion i.e.* $\limsup_{s\to\pm\infty} |\gamma(s,0)| = \infty$*. Then* $\limsup_{s\to\pm\infty} |\gamma(s,t)| = \infty$ *for all $t \in \mathbb{R}$ so that each map $\gamma(\cdot,t)$ is proper and thus ϕ is proper.*

Proof. For each $t \in \mathbb{R}$ since $|\gamma_t| \leq 1$ we have $|\gamma(s,t)| \geq |\gamma(s,0)| - \int_0^t |\gamma_t(s,\tilde{t})| d\tilde{t} \geq |\gamma(s,0)| - t$ so $\limsup_{s\to\pm\infty} |\gamma(s,t)| = \infty$ for all $t \in \mathbb{R}$ as claimed. $\square$

Let us now observe that Σ_{sing} is singular, at least in the sense that it consists of null points. A similar observation to the following was made, as part of a broader context, in [42, Theorem 3.1].

Lemma 4.2. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C, V) and suppose $\mathcal{K}$ as defined in (4.2) is non-empty. Suppose that for some neighbourhood U of a point $q \in \partial\mathcal{K}$ there exists a C^1 embedded surface $\mathcal{T} \subseteq \mathbb{R}^{1+2}$ such that $\phi(U) \subseteq \mathcal{T}$. Then $\mathcal{T}$ is null at $\phi(q)$.*

Proof. Suppose that U is a neighbourhood of $q \in \partial\mathcal{K}$ such that $\phi(U) \subseteq \mathcal{T}$ where $\mathcal{T}$ is some C^1 embedded surface. For each point $(s,t) \in U \setminus \mathcal{K}$ the tangent space $T_{\phi(s,t)}\mathcal{T}$ is a timelike plane which intersects the light cone along null directions spanned by the nowhere vanishing null vectors

$$\phi_s(s,t) + \phi_t(s,t) = A_+(s+t) = (1, a_+(s+t))$$

and

$$\phi_s(s,t) - \phi_t(s,t) = A_-(s-t) = (1, a_-(s-t)).$$

Choose a sequence of points $(s_k, t_k) \in U \setminus \mathcal{K}$ with $(s_k, t_k) \to q = (s_*, t_*)$. Since $a_+(s_* + t_*) = a_-(s_* - t_*)$ it follows that $\lim_{(s_k,t_k)\to(s_*,t_*)} a_+(s_k + t_k) = \lim_{(s_k,t_k)\to(s_*,t_*)} a_-(s_k - t_k)$, so the null lines along which $T_{\phi(s_k,t_k)}\mathcal{T}$ intersects the light cone converge. So $T_{\phi(s_*,t_*)}\mathcal{T} = \lim_{(s_k,t_k)\to(s_*,t_*)} T_{\phi(s_k,t_k)}\mathcal{T}$ must be a null plane. $\square$

The following result is an immediate consequence of Theorem 3.1.

Lemma 4.3. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be an evolution by isothermal gauge for a $C^2 \times C^1$ initial data (C, V) and suppose $\mathcal{K}$ as defined in (4.2) is non-empty. Suppose for* *some $q = (s_0, t_0) \in \mathcal{K}$ there exists $\varepsilon > 0$ and an open set $\Omega \subseteq \mathbb{R}^2 \setminus \mathcal{K}$ such that $\{s_0\} \times [t_0 - \varepsilon, t_0) \subseteq \Omega$. Then, writing $k(\cdot, t)$ for the curvature of the (planar) curves $\gamma(\cdot, t)$, we have*

$$\int_{t_0-\varepsilon}^{t_0} |k(s_0, t)| dt = \infty. \tag{4.3}$$

In particular, for every neighbourhood U of q, the set $\phi(U)$ is not a subset of any C^2 immersed surface in $\mathbb{R}^{1+2}$.

Proof. (4.3) follows immediately from Theorem 3.1. Suppose now for a contradiction that for some neighbourhood U of q there exists a C^2 immersed surface $\mathcal{T} \subseteq \mathbb{R}^{1+2}$ such that $\phi(U) \subseteq \mathcal{T}$. Let $\{V_i\}_{i \in I}$ be an open cover of $\mathcal{T}$ by C^2 embedded surfaces, and choose $j \in I$ such that $q \in \phi^{-1}(V_j)$. Then defining $U_j = \phi^{-1}(V_j)$, we have a neighbourhood U_j of q such that $\phi(U_j) = V_j$ is a C^2 embedded surface. But then by Lemma 4.2 V_j is null at $\phi(q)$ so by the implicit function theorem there exists a C^2 parametrization of V_j in some neighbourhood of $\phi(q)$ of the form $(s, t) \mapsto (t, \psi(s, t))$. But that implies the cross sections $\psi(\cdot, t)$ have uniformly bounded curvatures $k(\cdot, t)$ in a neighbourhood of q, in contradiction with (4.3). $\square$

4.2 Some analysis of singular points

Let (C, V) be a $C^k \times C^{k-1}$ initial data parametrized isothermally. Write

$$U_0(s) = \frac{c'(s)}{|c'(s)|} \in S^1 \subseteq \mathbb{R}^2 \tag{4.4}$$

for the unit tangent map along C. Let $\vartheta \colon \mathbb{R} \to \mathbb{R}$ be a lift of $U_0 \colon \mathbb{R} \to S^1$, so that

$$U_0(s) = (\cos \vartheta(s), \sin \vartheta(s)). \tag{4.5}$$

If C is C^2 then ϑ is related to the curvature k of C by

$$\int_{s_1}^{s_2} k(s)dl(s) = \vartheta(s_2) - \vartheta(s_1)$$

where $dl(s) = |c'(s)|ds$ is the element of arclength along C.

Since $\langle c'(s), v(s)\rangle = 0$ we may define a function

$$\mu \colon \mathbb{R} \to (-1, 1)$$

such that

$$v(s) = \mu(s)U_0(s)^{\perp} = \mu(s)(-\sin\vartheta(s), \cos\vartheta(s)). \tag{4.6}$$

Next, recalling the definition $a_{\pm}(s) = v(s) \pm c'(s)$, by trigonometric identities it may be seen that the quantities

$$\alpha_+(s) = \vartheta(s) + \arcsin(\mu(s)) \tag{4.7}$$

$$\alpha_-(s) = \vartheta(s) - \arcsin(\mu(s)) - \pi \tag{4.8}$$

define a pair of lifts for $a_{\pm}$, so that

$$a_{\pm}(s) = (\cos\alpha_{\pm}(s), \sin\alpha_{\pm}(s)),$$

see Figure 4.1.

Remark 4.4. The function μ defined by (4.6) may be given a geometric interpretation as follows. Defining $\varphi(s) = \operatorname{arctanh}\mu(s)$ we see that

$$N(s) = (\sinh\varphi(s), -\cosh\varphi(s)\sin\vartheta(s), \cosh\varphi(s)\cos\vartheta(s))$$

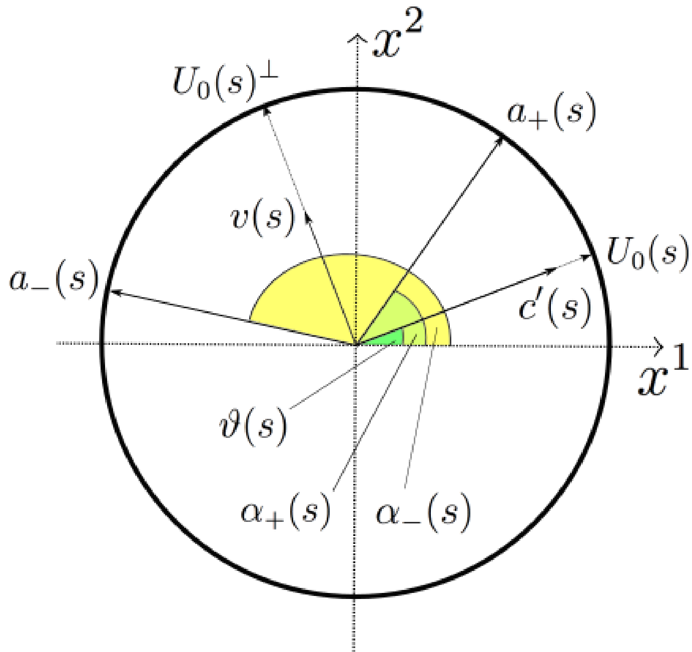


Figure 4.1: The isothermal frame in angular coordinates.

defines a spacelike unit normal to $T_{C(s)}\Sigma = \mathrm{span}\{C'(s), V(s)\}$. The angles (ϑ, φ) may be thought of as longitude-latitude coordinates on the 1-sheeted hyperboloid $S^{1+1} = \{-(x^0)^2 + (x^1)^2 + (x^2)^2 = 1\} \subseteq \mathbb{R}^{1+2}$ (i.e. the de Sitter sphere).

Denote the characteristic diamond associated to the interval $[s_1, s_2]$ by

$$D(s_1, s_2) = \Big\{(s,t) \in \mathbb{R}^2 \colon s_1 + |t| \leq s \leq s_2 - |t|\Big\}. \tag{4.9}$$

The following result is a localized singularity statement for C^1 evolutions by isothermal gauge (compare with Theorem 2.3).

Proposition 4.5. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C, V). Writing U_0 for the unit tangent along C as in (4.4), suppose that $\mathrm{Im}(U_0)$ contains a closed semi-circle i.e. suppose there exist $s_1 < s_2$ such* *that*

$$|\vartheta(s_2) - \vartheta(s_1)| \geq \pi$$

where ϑ *is as in* (4.5)*. Then, with* $\mathcal{K}$ *as in* (4.2) *and* $D(s_1, s_2)$ *as in* (4.9)*, it follows that* $\mathcal{K} \cap D(s_1, s_2)$ *is non-empty.*

Remark 4.6. Note that the same conclusion cannot be reached if $\mathrm{Im}(U_0)$ contains only a half-closed semi-circle. Indeed, in Example 2.9 we had $\mathrm{Im}(\vartheta) = [-\frac{\pi}{2}, \frac{\pi}{2})$ whilst $\mathcal{K} = \emptyset$.

Proof of Proposition 4.5: Identities (4.7) and (4.8) give

$$|(\alpha_+(s_2) - \alpha_+(s_1)) + (\alpha_-(s_2) - \alpha_-(s_1))| = 2\,|\vartheta(s_2) - \vartheta(s_1)| \geq 2\pi$$

thus $a_+([s_1, s_2])$ and $a_-([s_1, s_2])$ cannot form disjoint subsets of S^1 and so there exist $\xi, \eta \in [s_1, s_2]$ such that $a_+(\xi) = a_-(\eta)$. $\square$

Note that Proposition 4.5 applies immediately to the case of a self-intersecting curve C thanks to the following elementary result.

Lemma 4.7. *Suppose* $c\colon \mathbb{R} \to \mathbb{R}^2$ *is a* C^1 *immersion with a point of self-intersection i.e. suppose there exist* $r_1 < r_2$ *such that* $c(r_1) = c(r_2)$*. Let* U_0 *denote the unit tangent along* c *as in* (4.4)*. Then* $U_0([r_1, r_2])$ *contains an arc of length* $> \pi$*.*

Proof. Since $c(r_1) = c(r_2)$, we have

$$\int_{r_1}^{r_2} \langle c'(s), \omega \rangle ds = 0 \tag{4.10}$$

for every $\omega \in \mathbb{R}^2$. But we now claim that if $U_0([r_1, r_2])$ is contained in a closed semi-circle, then there exists an $\omega_0 \in \mathbb{R}^2$ such that

$$\int_{r_1}^{r_2} \langle c'(s), \omega_0 \rangle ds > 0. \tag{4.11}$$

Indeed, to show (4.11), first note that if $U_0([r_1, r_2]) = \{p\}$ is a single point, then (4.11) clearly holds with $\omega_0 = p$, so we may assume $U_0([r_1, r_2])$ is not a single point. Next, by rotating coordinates, take $U_0([r_1, r_2])$ to be contained in the closed upper semi circle $S^1_+ = \{(x^1, x^2)\colon x^2 = \sqrt{1-(x^1)^2} \geq 0\}$. Then taking $\omega_0 = (0, 1)$ gives $\langle c'(s), \omega_0 \rangle \geq 0$ for all $s \in [r_1, r_2]$ and $\langle c'(s_0), \omega_0 \rangle > 0$ for some $s_0 \in [r_1, r_2]$, and (4.11) follows. Since (4.10) and (4.11) are incompatible, we conclude that $U_0([r_1, r_2])$ contains an arc of length $> \pi$ as claimed. $\square$

Remark 4.8 (A self-intersecting initial data for which only the past evolution by isothermal gauge is singular)**.** Proposition 4.5 states that if $U_0(s_1, s_2)$ contains a closed semi-circle then $\mathcal{K} \cap D(s_1, s_2)$ is non-empty. But this does not tell us whether singularity occurs for $t > 0$ or $t < 0$ (i.e. whether singularity is in the future or the past Cauchy evolution of (C, V)). In fact, it is a simple exercise to cook up an initial data (C, V) for which the curve C is self-intersecting and for which $a_+(\xi) \neq a_-(\eta)$ for all $\xi \geq \eta$ so that $\mathcal{K} \cap \{t \geq 0\} = \emptyset$. An example of such an initial data is sketched in Figure 4.2.

Proposition 4.5 gives a sufficient condition in terms of ϑ for $\mathcal{K} \cap D(s_1, s_2)$ to be non-empty. We can also give a sufficient condition for no singularity in terms of ϑ and the initial velocity v.

Lemma 4.9. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C, V) which is parametrized so that $V(s) = (1, v(s))$ where $\langle c'(s), v(s) \rangle =$*

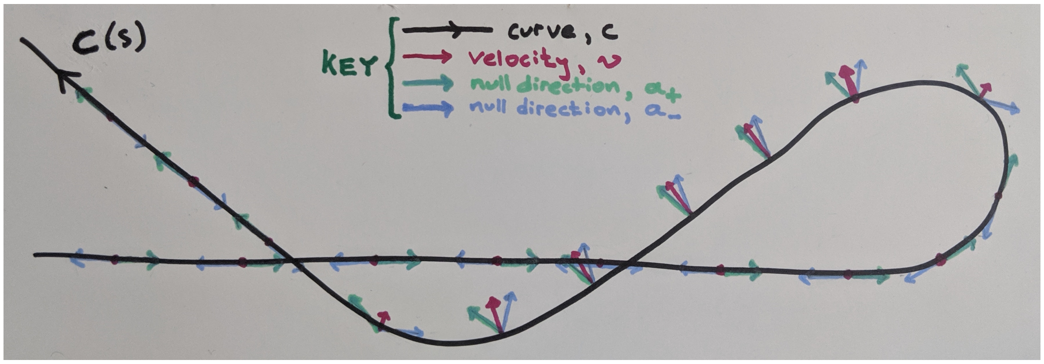


Figure 4.2: A sketch of an initial data (C, V) for which $C(s) = (0, c(s))$ is a self-intersecting curve and $V(s) = (1, v(s))$ is chosen (with $v(s) = 0$ outside of a compact subset) such that $a_+(\xi) \neq a_-(\eta)$ for all $\xi \geq \eta$ (so that the future evolution by isothermal gauge of (C, V) is non-singular).

0. *Writing* ϑ *as in* (4.5)*, suppose that*

$$\sup_{r_1, r_2 \in [s_1, s_2]} |\vartheta(r_2) - \vartheta(r_1)|^2 + \sup_{r \in [s_1, s_2]} |v(r)|^2 < 1. \tag{4.12}$$

Then with $\mathcal{K}$ *as in* (4.2) *and* $D(s_1, s_2)$ *as in* (4.9)*, it follows that*

$$\mathcal{K} \cap D(s_1, s_2) = \emptyset. \tag{4.13}$$

Proof. Writing $a_\pm$ as in (2.19) it follows easily from (4.12) and trigonometric identities that $a_+(\xi) \neq a_-(\eta)$ for $\xi, \eta \in [s_1, s_2]$ (refer to Figure 4.1). The claim follows. □

Remark 4.10 (Small data global existence)**.** If the initial data (C, V) satisfies the estimate (4.12) on $[s_1, s_2] = \mathbb{R}$ then by Lemma 4.9 it follows that $\mathcal{K} = \emptyset$ so the evolution by isothermal gauge ϕ parameterises a properly immersed timelike maximal surface $\Sigma = \mathrm{Im}(\phi)$ which contains C and is tangent to V along C. This is a global existence result which does not require any decay of initial data (C, V) at infinity, and may be compared with the interesting recent results of [53] and [74].

Corollary 4.11. *Let* $C \colon \mathbb{R} \to \mathbb{R}^{1+2}$ *be given as* $C(s) = (0, s, 0)$ *(i.e.* $\mathrm{Im}(C)$ *is a* *straight line), let V be any smooth timelike velocity along C and let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be the evolution of (C, V) by isothermal gauge. Then $\mathcal{K} = \emptyset$ (so ϕ is an immersion and $\Sigma = \mathrm{Im}(\phi)$ is a smooth properly immersed timelike maximal surface containing $\mathrm{Im}(C)$ and tangent to V along C).*

Proof. Since $\vartheta \equiv 0$ and V is timelike, estimate (4.13) holds on the interval $[s_1, s_2] = \mathbb{R}$, so $\mathcal{K} = \emptyset$ by Lemma 4.9. $\square$

Remark 4.12. If $C\colon \mathbb{R} \to \mathbb{R}^{1+2}$ is a smooth proper immersion such that $\mathrm{Im}(C)$ is not a straight line, then it is easy to find a smooth vector field V along C for which $\mathcal{K} \neq \emptyset$ (i.e. the evolution ϕ of (C, V) in isothermal gauge becomes singular in finite time). Indeed, let $C(s) = (0, c(s))$ and $U_0(s) = {}^{c'(s)}/_{|c'(s)|}$, and choose $s_1, s_2 \in \mathbb{R}$ so that $U_0(s_1) \neq U_0(s_2)$. Let $\beta \in (0, 2\pi)$ be such that $U_0(s_2)$ is given by an anti-clockwise rotation of $U_0(s_1)$ in the plane by β radians and define $V(s) = (1, v(s))$ by $v(s) = \cos\frac{\beta}{2} U_0(s)^{\perp}$ where $\perp$ denotes anti-clockwise rotation in the plane by $\frac{\pi}{2}$ radians. Writing $a_{\pm}(s) = v(s) \pm \sin\frac{\beta}{2} U_0(s)$ for the spatial components of the null vectors $A_{\pm}(s) = (1, a_{\pm}(s))$ which span the tangent plane $T_{C(s)}\mathrm{Im}(\phi) = \mathrm{span}\{C'(s), V(s)\}$, we may compute from the trigonometric identities (4.7) and (4.8) that $a_+(s_2) = a_-(s_1)$, so $\mathcal{K} \neq \emptyset$.

From Lemma 4.9 we obtain the following short-time existence result, which does not require any decay of the initial data at infinity.

Corollary 4.13 (Short-time existence)**.** *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be an evolution by isothermal gauge for a $C^k \times C^{k-1}$ initial data (C, V) ($k \geq 1$) and let U_0 denote the unit tangent vector along C as in* (4.4)*. Suppose that U_0 is uniformly continuous and V is uniformly timelike (i.e. with $V = (1, v)$ we have $\sup_{s\in\mathbb{R}} |v(s)| < 1$). Then there exists $T > 0$ depending only on $\sup_{s\in\mathbb{R}} \frac{1}{1-|v(s)|}$ and the modulus of continuity of U_0 such that $\mathcal{K} \cap \{|t| \leq T\} = \emptyset$ (so $\mathrm{Im}(\phi) \cap \{(x^0, x^1, x^2)\colon |x^0| \leq T\}$ is a C^k immersed timelike maximal surface containing $\mathrm{Im}(C)$ and tangent to V along C.)*

Proof. Take $\varepsilon > 0$ so that $\sup_{s\in\mathbb{R}} |v(s)|^2 \leq 1-\varepsilon$. Since U_0 is uniformly continuous there exists $\delta > 0$, depending only on ε and the modulus of continuity of U_0, such that $|\vartheta(r_2) - \vartheta(r_1)|^2 < \varepsilon$ provided $|r_1 - r_2| \leq \delta$. Defining $s_k = \frac{\delta k}{2}$ for all $k \in \mathbb{Z}$ gives

$$\sup_{r_1, r_2 \in [s_k, s_{k+2}]} |\vartheta(r_2) - \vartheta(r_1)|^2 + \sup_{r \in [s_k, s_{k+2}]} |v(r)|^2 < 1,$$

so $\mathcal{K} \cap D(s_k, s_{k+2}) = \emptyset$ for all $k \in \mathbb{Z}$ by Lemma 4.9. With $T := \frac{\delta}{4}$, the set $\{|t| \leq T\}$ is then contained in $\cup_{k\in\mathbb{Z}} D(s_k, s_{k+2})$, so $\mathcal{K} \cap \{|t| \leq T\} = \emptyset$ as claimed. $\square$

4.3 Examples of C^1 properly embedded surfaces which are smooth timelike maximal surfaces away from some null set

We will now give some (non-generic) examples of smooth initial data for which the Cauchy evolution for a timelike maximal surface becomes singular in finite time, but the evolution by isothermal gauge beyond singular time yields a C^1 embedded surface.

Example 4.14 (C^1 embedded surfaces which are smooth timelike maximal surfaces away from a pair of null lines and contain non-graphical compact sets)**.** Let l_1 and l_2 be the parallel half lines which take their endpoints at $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ and which are obtained as left and right translations respectively by a distance $\frac{1}{2}$ of the upper x^2-axis. Let f be a smooth segment of embedded curve of length $2L > 0$ which smoothly joins l_1 and l_2 at their endpoints, such that the unit tangent along f has non-vanishing x^1 component everywhere except at the endpoints. See Figure 4.3(a) for a C^1 approximation of such a curve. Let $c\colon \mathbb{R} \to \mathbb{R}^2$ be a parameterisation of

$l_1 \cup l_2 \cup f$ by arclength

$$c(s) = \begin{cases} \left(-\frac{1}{2}, -s-L\right) & \text{for } s \in (-\infty, -L] \\ (f^1(s), f^2(s)) & \text{for } s \in (-L, L) \\ \left(\frac{1}{2}, s-L\right) & \text{for } s \in [L, \infty). \end{cases}$$

Writing $c'(s) = (\cos\vartheta(s), \sin\vartheta(s))$, and we see $\mathrm{Im}(\vartheta) = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, $\vartheta(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $s \in (-L, L)$. The evolution of $C(s) = (0, c(s))$ with initial velocity $V = (1,0,0)$ by isothermal gauge is $\phi(s,t) = (t, \gamma(s,t))$, where $\gamma(s,t) = \frac{1}{2}\left(c(s+t) + c(s-t)\right)$. By Proposition 4.5, it follows that $\mathcal{K}$ (as defined in (4.2)) is non-empty.

Let us compute Σ_{sing}. Since $c'(s+t) = -c'(s-t)$ if and only if $s-t \leq -L$ whilst $s+t \geq L$ or $s-t \geq L$ whilst $s+t \leq -L$, it follows $\mathcal{K} = \mathcal{K}^+ \cup \mathcal{K}^-$ where

$$\mathcal{K}^+ = \left\{(s,t) \colon t \geq L,\ L-t \leq s \leq t-L\right\},$$
$$\mathcal{K}^- = \left\{(s,t) \colon t \leq -L,\ t+L \leq s \leq -t-L\right\}.$$

We then see $\Sigma_{\text{sing}} = \Sigma^+_{\text{sing}} \cup \Sigma^-_{\text{sing}}$, where

$$\Sigma^+_{\text{sing}} = \left\{(t, 0, t-L) \colon t \geq L\right\},$$
$$\Sigma^-_{\text{sing}} = \left\{(t, 0, -t-L) \colon t \leq -L\right\}.$$

i.e. Σ_{sing} consists of a pair of null half-lines, one emanating towards the future from the point $(L,0,0)$ and one emanating towards the past from the point $(-L,0,0)$. We have that $\Sigma \setminus \Sigma_{\text{sing}}$ is a smooth embedded timelike maximal surface. See Figure 4.3(b) for a numerical approximation of such a surface.

Note that the unit tangent $c'(s)$ is always confined to a closed semi-circle as

$(c')^1(s) \geq 0$. Writing $U(s,t) = \frac{\gamma_s(s,t)}{|\gamma_s(s,t)|} = \frac{c'(s+t)+c'(s-t)}{|c'(s+t)+c'(s-t)|}$ for the spatial unit tangent, defined only for $(s,t) \in \mathbb{R}^2 \setminus \mathcal{K}$, it is seen that $\lim_{(s,t)\to\mathcal{K}} U(s,t) = (1,0)$. Thus $U(s,t)$ extends continuously to a unit tangent vector field along $\gamma(s,t)$ and Σ is a C^1 embedded causal surface. Applying Theorem 3.1, we see that the curvature of the cross sections $\gamma(\cdot,t)$ blows up as $t \uparrow L$ so Σ is not a C^2 surface in any neighbourhood of $\phi(0,L)$. Since $\gamma(s,t) = \gamma(s,-t)$, we see that Σ is invariant under a reflection through the $\{x^0 = 0\}$ plane and so Σ is also not a C^2 surface in any neighbourhood of $\phi(0,-L)$. There exists a compact subset $K \subseteq \Sigma$ which is not a graph. We observe that the image of the spacelike unit normal in this example (defined only on $\Sigma \setminus \Sigma_{\text{sing}}$) intersects both connected boundary components of a closed hemi-hyperboloid.

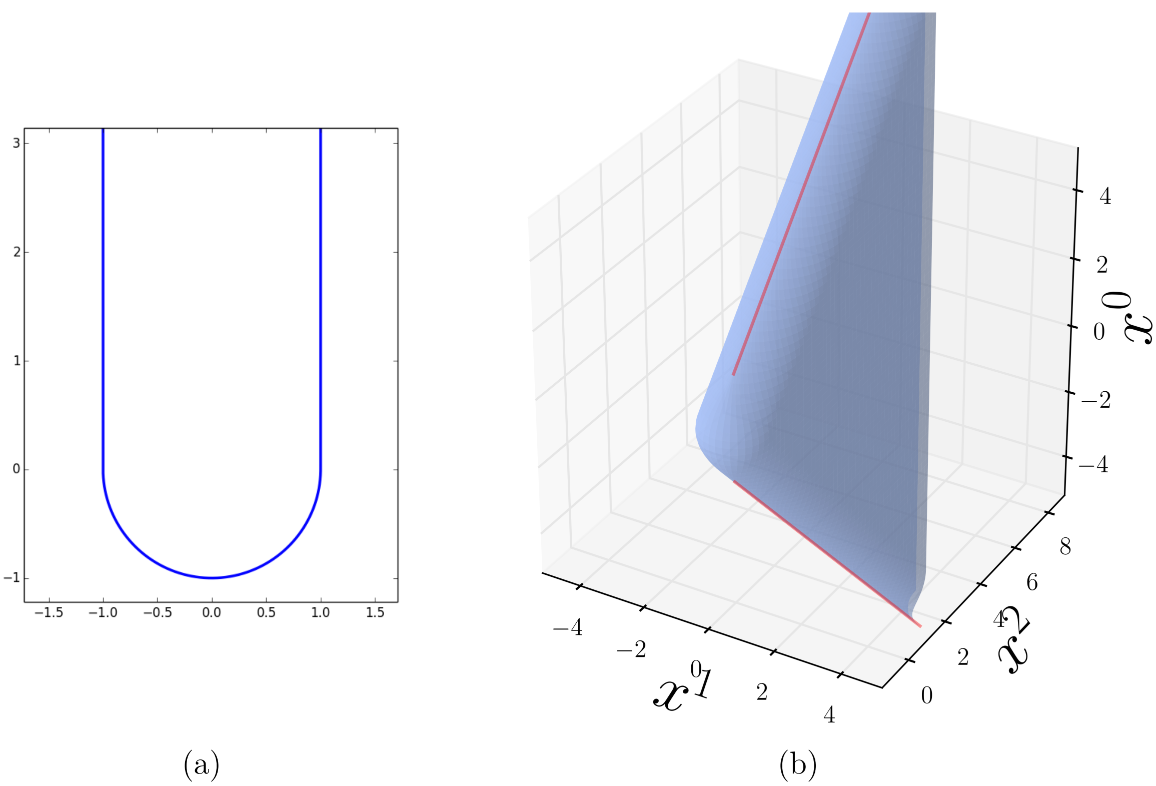


Figure 4.3: (a) A "cigar curve" which contains a compact subset which is not a graph. (b) Evolution of (a) by isothermal gauge to a C^1 embedded surface Σ which is a timelike maximal surface away from null lines Σ_{sing} shown in red. There is a compact subset $K \subseteq \Sigma$ which is not a graph.

Example 4.15 (C^1 embedded doubly-periodic surfaces which are smooth away from isolated null points situated on a rectangular lattice and which are graphs, but not C^1 graphs)**.** Let $f = (f^1, f^2)\colon [0, L] \to \mathbb{R}^2$ parametrize a section of curve by arclength so that $f(0) = (-1, 0)$, $f(L) = (1, 0)$, $\dot{f}^1(s) > 0$ for $s \in (0, L)$, $\dot{f}(0) = (0, 1)$, $\dot{f}(L) = (0, -1)$ and $\frac{d^k f^2}{ds^k}(0) = \frac{d^k f^2}{ds^k}(L) = 0$ for $k \geq 2$. Extend f periodically to a smooth immersion $c\colon \mathbb{R} \to \mathbb{R}^2$ by

$$c(s) = \begin{cases} (f^1(s), f^2(s)) & \text{for } s \in [0, L] \\ (2 + f^1(s), -f^2(s)) & \text{for } s \in (L, 2L) \\ (4n, 0) + c(s - 2nL) & \text{for } s \in [2nL, 2(n+1)L), n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

See Figure 4.4(a). It may be seen that $\mathrm{Im}(c)$ defines a graph over the x^1 axis, but not a C^1 graph. As c is parametrized by arclength, the evolution by isothermal gauge $\phi(s, t) = (t, \gamma(s, t))$ of the curve $C(s) = (0, c(s))$ with initial velocity $V = (1, 0, 0)$ is given by $\gamma(s, t) = \frac{1}{2}\left(c(s + t) + c(s - t)\right)$. Let us compute Σ_{sing}. Note that $(s, t) \in \mathcal{K}$ iff $\frac{s+t}{L}$ is an odd integer and $\frac{s-t}{L}$ is an even integer or vise-versa. From this we deduce that

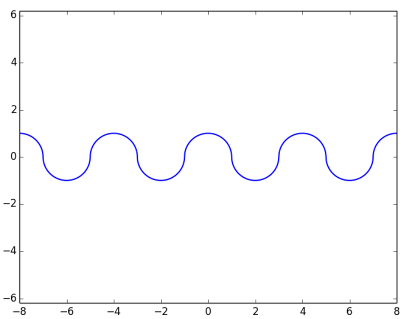
$$\mathcal{K} = \left\{ \left(\frac{mL}{2}, \frac{nL}{2} \right) : m \,\text{and}\, n \,\text{are odd integers} \right\}$$

and since $c(\frac{nL}{2}) = (n - 1, 0)$ for all $n \in \mathbb{Z}$, we have

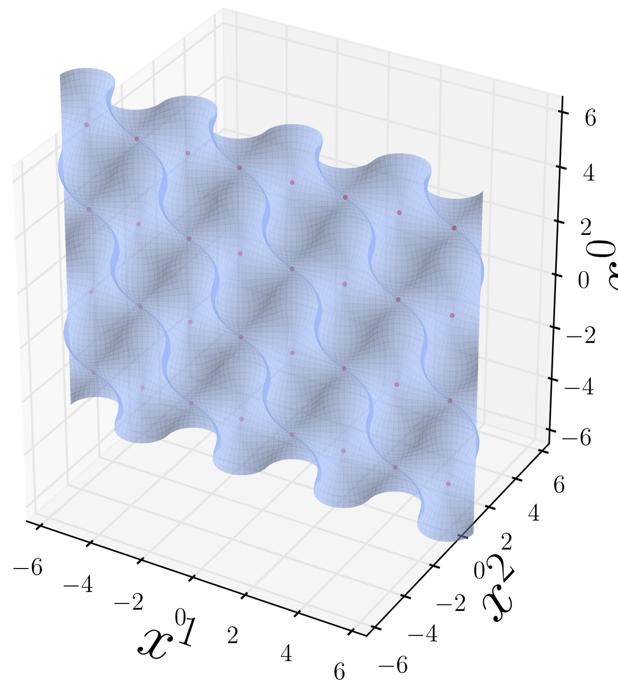
$$\Sigma_{\text{sing}} = \left\{ \left(\frac{nL}{2}, k, 0 \right) : n \text{ is an odd integer and } k \text{ is an even integer} \right\}$$

which is a rectangular lattice of isolated points. Σ is a smooth, timelike embedded surface away from Σ_{sing} and we observe that $(c')^1(s) \geq 0$ so $\lim_{(s,t)\to\mathcal{K}} U(s, t) = (1, 0)$ and Σ is a C^1 embedded causal surface. By Theorem 3.1 we see that Σ is not a C^2

surface in any neighbourhood of any point in Σ_{sing}. Σ is a graph over the x^0–x^1 plane, but not a C^1 graph. See Figure 4.4(b).



(a)



(b)

Figure 4.4: (a) A periodic wedge of hemi-circles which is a graph, but not a C^1 graph. (b) Evolution of (a) to a C^1 maximal surface which is a timelike maximal surface away from null points Σ_{sing} on a rectangular lattice shown in red. Σ is a graph over the x^0–x^1 plane, but not a C^1 graph.

4.4 Discontinuity of the spatial unit tangent

In both of Examples 4.14 and 4.15 note that $\mathrm{Im}(U_0)$ is exactly a closed semi-circle. In this section we will show that the behaviour observed in Examples 4.14 and 4.15 is 'borderline'. To be precise, we prove

Theorem 4.16. *Let (C, V) be a $C^1 \times C^0$ initial data where $C\colon \mathbb{R} \to \mathbb{R}^{1+2}$ is a proper immersion and let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$, $\phi(s,t) = (t, \gamma(s,t))$ be the evolution of (C, V) by* *isothermal gauge. Writing $U_0\colon \mathbb{R} \to S^1$ for the unit tangent vector along the initial curve $\gamma(\cdot, 0)$, suppose that $\mathrm{Im}(U_0)$ contains an arc of length $> \pi$. Then there exists an open interval $I \subseteq \mathbb{R}$ such that for every $t_* \in I$ either $\mathrm{Im}(\gamma(\cdot, t_*))$ is not a C^1 immersed curve, or $\mathrm{Im}(\gamma(\cdot, t_*))$ is a C^1 immersed curve but the spatial unit tangent $U(\cdot, t_*) = \gamma_s(\cdot,t_*)/|\gamma_s(\cdot,t_*)|$ (defined only on the set $\{s\colon \gamma_s(s, t_*) \neq 0\}$) admits no extension to a continuous unit tangent vector field along $\gamma(\cdot, t_*)$.*

Let (C, V) be a $C^1 \times C^0$ initial data parameterised isothermally and let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$, $\phi(s, t) = (t, \gamma(s, t))$ be the evolution of (C, V) by isothermal gauge. As in §4.2 we write $a_\pm(s) = v(s) \pm c'(s)$ so that $|a_\pm(s)|^2 = 1$. Recall from (4.7)–(4.8) that $a_\pm(s) = (\cos \alpha_\pm(s), \sin \alpha_\pm(s))$ where

$$\alpha_+(s) = \vartheta(s) + \arcsin(\mu(s))$$
$$\alpha_-(s) = \vartheta(s) - \arcsin(\mu(s)) - \pi,$$

where ϑ and μ are defined by (4.5) and (4.6). Let us now introduce

$$\beta(s, t) = \alpha_+(s + t) - \alpha_-(s - t) \tag{4.14}$$

so that $\beta(s, t)$ is the angle between the null directions $a_+(s + t)$ and $a_-(s - t)$. Note that we have

$$\beta(s, 0) = \alpha_+(s) - \alpha_-(s) = 2 \arcsin(\mu(s)) + \pi \in (0, 2\pi) \tag{4.15}$$

for all $s \in \mathbb{R}$.

The proof of Theorem 4.16 will be via a study of the spatial unit tangent map

$$U(s, t) = \frac{\gamma_s(s, t)}{|\gamma_s(s, t)|},$$

which is well defined for $(s,t) \in \mathbb{R}^2 \setminus \mathcal{K}$. From (4.1) one may compute explicitly

$$U(s,t) = \operatorname{sgn}\left(\sin\frac{\beta(s,t)}{2}\right) e(s,t) \tag{4.16}$$

where

$$e(s,t) = \left(-\sin\frac{\alpha_+(s+t)+\alpha_-(s-t)}{2}, \cos\frac{\alpha_+(s+t)+\alpha_-(s-t)}{2}\right) \tag{4.17}$$

is a continuous unit vector field along $\gamma(s,t)$ (note that $s \mapsto e(s,t)$ does not necessarily define a unit tangent vector field along $s \mapsto \gamma(s,t)$).

We have $(s,t) \in \mathcal{K}$ precisely when $\beta(s,t) \in 2\pi\mathbb{Z}$. From formula (4.16) it is apparent that to study when U becomes discontinuous requires an analysis of when $\sin\left(\frac{\beta}{2}\right)$ changes sign.

Lemma 4.17. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C, V). With U_0 denoting the unit tangent along C as in* (4.4) *suppose that $\operatorname{Im}(U_0)$ contains an arc of length $> \pi$, i.e. suppose there exist $s_1, s_2 \in \mathbb{R}$ such that*

$$\vartheta(s_2) - \vartheta(s_1) > \pi \tag{4.18}$$

where ϑ is as in (4.5)*. Then, with β as in* (4.14)*, there exists $(s_*, t_*) \in \mathbb{R}^2$ such that $\sin\left(\frac{\beta(s_*,t_*)}{2}\right) < 0$. Furthermore, if $C\colon \mathbb{R} \to \{x^0 = 0\} \subseteq \mathbb{R}^{1+2}$ is a proper immersion, then there exists a time $t_* \in \mathbb{R}$ such that $\sin\left(\frac{\beta(\cdot,t_*)}{2}\right)$ takes both positive and negative values.*

Proof. By identities (4.7) and (4.8), we have

$$\Big(\alpha_+(s_2) - \alpha_-(s_1)\Big) - \Big(\alpha_+(s_1) - \alpha_-(s_2)\Big) = 2\Big(\vartheta(s_2) - \vartheta(s_1)\Big) > 2\pi$$

and so, setting $s_0 = \frac{1}{2}(s_1 + s_2)$ and $t_0 = \frac{1}{2}(s_1 - s_2)$ gives

$$\begin{aligned}\beta(s_0, -t_0) - \beta(s_0, t_0) &= \Big(\alpha_+(s_0 - t_0) - \alpha_-(s_0 + t_0)\Big) - \Big(\alpha_+(s_0 + t_0) - \alpha_-(s_0 - t_0)\Big) \\ &> 2\pi.\end{aligned}$$

It follows that one of $\beta(s_0, -t_0) > 2\pi$ or $\beta(s_0, t_0) < 0$ holds, and since β is continuous and $\beta(\cdot, 0) \in (0, 2\pi)$ by (4.15), it follows that $\sin\left(\frac{\beta(s_*, t_*)}{2}\right) < 0$ for some (s_*, t_*) as claimed.

Now suppose in addition that C is proper and suppose for a contradiction that there exists no time $t_* \in \mathbb{R}$ such that $\sin\left(\frac{\beta(\cdot, t_*)}{2}\right)$ takes both positive and negative values. Write

$$\begin{aligned}A &= \{t \in \mathbb{R} \colon \sin \frac{\beta(s,t)}{2} \geq 0 \, \text{for all} \, s \in \mathbb{R}\} \\ B &= \{t \in \mathbb{R} \colon \sin \frac{\beta(s,t)}{2} \leq 0 \, \text{for all} \, s \in \mathbb{R}\}.\end{aligned}$$

Then A and B are closed sets, and we are supposing that $A \cup B = \mathbb{R}$. Note that A is non-empty by (4.15), whilst B is non-empty by the argument of the previous paragraph, and so by connectedness of $\mathbb{R}$, $A \cap B$ must be non-empty. Taking $t_1 \in A \cap B$ gives $\beta(\cdot, t_1) \equiv 2k\pi$ for some $k \in \mathbb{Z}$, which implies $\gamma_s(\cdot, t_1) \equiv 0$ so $\mathrm{Im}(\gamma(\cdot, t_1))$ consists of a single point. But since C is proper this contradicts Lemma 4.1. The lemma is proved. $\square$

We will deduce Theorem 4.16 from Lemma 4.17 together with the following.

Lemma 4.18. *Let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$, $\phi(s,t) = (t, \gamma(s,t))$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C, V) and let β be as in (4.14). Suppose there exists $t_* \in \mathbb{R}$ such that $\sin\left(\frac{\beta(\cdot, t_*)}{2}\right)$ takes both positive and negative values on an interval $[s_1, s_2]$. Then for any $\zeta > 0$, there is an open interval $I \subseteq (t_* - \zeta, t_* + \zeta)$, such that* *for all $t \in I$ either $\gamma([s_1, s_2], t)$ is not a C^1 immersed curve, or $\gamma([s_1, s_2], t)$ is a C^1 immersed curve but $U(\cdot, t) = \gamma_s(\cdot,t)/|\gamma_s(\cdot,t)|$ admits no continuous extension to a unit tangent vector field along $\gamma(\cdot, t)$ on $[s_1, s_2]$.*

Proof. We will follow the proof of [42, Theorem 5.1(iii)]. We will assume that $[s_1, s_2]$ is such that $\beta(s_1, t_*) < 0$ and $\beta(s_2, t_*) > 0$, as all other cases may be treated analogously.

By continuity there exists $\delta_0 \in (0, \zeta]$ such that $\beta(s_1, t) < 0$ and $\beta(s_2, t) > 0$ for all $t \in (t_* - \delta_0, t_* + \delta_0)$. Suppose for some $t_0 \in (t_* - \delta_0, t_* + \delta_0)$ we have that $\gamma([s_1, s_2], t_0)$ is a C^1 immersed curve and $U(\cdot, t_0)$ extends to a continuous unit tangent vector field $\hat{U}(\cdot, t_0)$ along $\gamma(\cdot, t_0)$ on the interval $[s_1, s_2]$ (we will see such a situation in Example 4.19). Define

$$\begin{aligned} r_2 &= \sup\{\hat{s} \in [s_1, s_2] \colon \beta(s, t_0) \le 0 \quad \text{for all}\, s \in [s_1, \hat{s}]\}, \\ r_1 &= \inf\{\hat{s} \in [s_1, r_2] \colon \beta(s, t_0) = 0 \quad \text{for all}\, s \in [\hat{s}, r_2]\}, \end{aligned} \tag{4.19}$$

then

$$\beta(s, t_0) = 0 \quad \text{for all}\, s \in [r_1, r_2] \tag{4.20}$$

and β takes both positive and negative values in every neighbourhood of $[r_1, r_2]$.

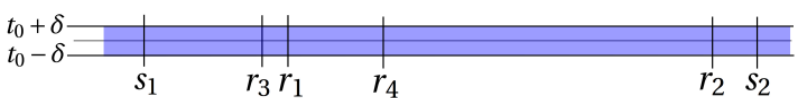


Figure 4.5: The terms in the proof of Lemma 4.18.

We claim that

$$\alpha_+(r_1 + t_0) = \alpha_+(r_2 + t_0) + m\pi \quad \text{for some odd integer}\, m. \tag{4.21}$$

To show (4.21), note that since $\gamma(s, t_0) = \gamma(r_1, t_0)$ for all $s \in [r_1, r_2]$ it follows that

$\hat{U}(r_1, t_0) = \hat{U}(r_2, t_0)$. Take sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to r_1$, $\beta(x_n, t_0) < 0$, and $y_n \to r_2$, $\beta(y_n, t_0) > 0$ (which is possible from the definitions of r_1 and r_2). Then from (4.16)

$$
\begin{aligned}
\hat{U}(r_1, t_0) &= \lim_{x_n \to r_1} \left\{ \operatorname{sgn} \left(\sin \frac{\beta(x_n, t_0)}{2} \right) \right\} e(r_1, t_0) \\
\hat{U}(r_2, t_0) &= \lim_{y_n \to r_2} \left\{ \operatorname{sgn} \left(\sin \frac{\beta(y_n, t_0)}{2} \right) \right\} e(r_2, t_0) \\
&= - \lim_{x_n \to r_1} \left\{ \operatorname{sgn} \left(\sin \frac{\beta(x_n, t_0)}{2} \right) \right\} e(r_2, t_0),
\end{aligned}
$$

so $e(r_1, t_*) = -e(r_2, t_*)$ from which (4.21) follows from (4.17) and (4.20).

Geometrically, (4.20) and (4.21) amount to the statement that the null directions $a_+(s + t_0)$ and $a_-(s - t_0)$ are aligned for $s \in [r_1, r_2]$ and undergo a rotation by a non-trivial multiple of π radians as s varies from r_1 to r_2. We will now show that this situation will be lost after a small perturbation of t_0. More precisely, we will show that for any $\varepsilon > 0$ there is an open interval I either of the form $I = (t_0, t_0 + \delta)$ or $I = (t_0 - \delta, t_0)$ for some $\delta > 0$ such that for each $t \in I$, there exists an interval $J = J(t) \subseteq [s_1, s_2]$ such that $\beta(\cdot, t)$ takes both positive and negative values on J and $|\alpha_+(w_1 + t) - \alpha_+(w_2 + t)| < \varepsilon$ for all $w_1, w_2 \in J$. Taking ε smaller than π, this will imply that condition (4.21) with t_0 replaced by t cannot hold for any $r_1, r_2 \in J$, so we will conclude that for each $t \in I$, the unit tangent $U(\cdot, t)$ admits no continuous extension to a unit tangent map, thus completing the proof of the lemma.

Fix $\varepsilon > 0$. By (4.19) and continuity of α_+ there exists $r_3 \in [s_1, r_1)$ such that $\beta(r_3, t_0) < 0$ and

$$
|\alpha_+(s + t_0) - \alpha_+(r_1 + t_0)| < \frac{\varepsilon}{4} \quad \text{for } s \in [r_3, r_1]. \tag{4.22}
$$

Take $\delta > 0$ so that

$$\beta(r_3, t) < 0 \quad \text{for } t \in [t_0 - \delta, t_0 + \delta]. \tag{4.23}$$

By the uniform continuity of α_+ on compact sets, by refining $\delta > 0$ to a smaller number as necessary, we may ensure

$$|\alpha_+(s + t) - \alpha_+(s + t_0)| < \frac{\varepsilon}{4} \quad \text{for } s \in [s_1, s_2], t \in [t_0 - \delta, t_0 + \delta]. \tag{4.24}$$

By (4.21), we can define

$$r_4 = \inf \left\{ s \in [r_1, r_2] \colon |\alpha_+(s + t_0) - \alpha_+(r_1 + t_0)| = \frac{\varepsilon}{4} \right\}. \tag{4.25}$$

We will first treat the case where $\alpha_+(r_4 + t_0) = \alpha_+(r_1 + t_0) + \frac{\varepsilon}{4}$. By refining $\delta > 0$ to be smaller as necessary, we may assume that $\alpha_+(w_2 + t_0) > \alpha_+(w_1 + t_0)$ provided $w_1 \in [r_1, r_1 + \delta]$ and $w_2 \in [r_4 - \delta, r_4]$. Then for each $\tau \in (0, \delta]$, we have

$$\begin{aligned} \int_{r_1}^{r_4 - \tau} (\alpha_+(s + \tau + t_0) - \alpha_+(s + t_0))ds &= \int_{r_4 - \tau}^{r_4} \alpha_+(s + t_0)ds - \int_{r_1}^{r_1 + \tau} \alpha_+(s + t_0)ds \\ &> 0 \end{aligned}$$

which shows that there exists an $s(\tau) \in [r_1, r_4 - \tau]$ such that $\alpha_+(s(\tau) + \tau + t_0) > \alpha_+(s(\tau) + t_0)$. We then see

$$\begin{aligned} \beta\left(s(\tau) + \frac{\tau}{2}, t_0 + \frac{\tau}{2}\right) &= \alpha_+(s(\tau) + \tau + t_0) - \alpha_+(s(\tau) - t_0) \\ &> \alpha_+(s(\tau) + t_0) - \alpha_+(s(\tau) - t_0) = \beta(s(\tau), t_0) \\ &\overset{(4.20)}{=} 0. \end{aligned} \tag{4.26}$$

Then for all $\tau \in (0, \delta]$, by (4.23) and (4.26) $\beta(\cdot, t_0 + \frac{\tau}{2})$ takes both positive and negative values on $J = J(t_0 + \frac{\tau}{2}) := [r_3, s(\tau) + \frac{\tau}{2}]$. On the other hand, for all $\omega_1, \omega_2 \in J$ we have

$$
\begin{aligned}
|\alpha_+(\omega_1 + t_0 + \frac{\tau}{2}) - \alpha_+(\omega_2 + t_0 + \frac{\tau}{2})| \leq{}& |\alpha_+(\omega_1 + t_0 + \frac{\tau}{2}) - \alpha_+(\omega_1 + t_0)| \\
&+|\alpha_+(\omega_2 + t_0 + \frac{\tau}{2}) - \alpha_+(\omega_2 + t_0)| + |\alpha_+(\omega_1 + t_0) - \alpha_+(r_1 + t_0)| \\
&+ |\alpha_+(\omega_2 + t_0) - \alpha_+(r_1 + t_0)|
\end{aligned}
$$

and since the first two terms on the right hand side of the above inequality are bounded by (4.24) and each of the last two terms is bounded by (4.22) and (4.25), this gives $|\alpha_+(\omega_1 + t_0 + \frac{\tau}{2}) - \alpha_+(\omega_2 + t_0 + \frac{\tau}{2})| < \varepsilon$ which is what we set out to show.

Next we treat the case where $\alpha_+(r_4 + t_0) = \alpha_+(r_1 + t_0) - \frac{\varepsilon}{4}$ by a similar argument. Choose $\delta > 0$ so that $\alpha_+(w_1 + t_0) > \alpha_+(w_2 + t_0)$ provided $w_1 \in [r_1, r_1 + \delta]$ and $w_2 \in [r_4 - \delta, r_4]$. Then, for all $\tau \in (0, \delta]$ there exists $s(\tau) \in [r_1 + \tau, r_4]$ such that $\alpha_+(s(\tau) - \tau + t_0) > \alpha_+(s(\tau) + t_0)$. In this case,

$$
\begin{aligned}
\beta\left(s(\tau) - \frac{\tau}{2}, t_0 - \frac{\tau}{2}\right) &= \alpha_+(s(\tau) - \tau + t_0) - \alpha_-(s(\tau) - t_0) \\
&> \alpha_+(s(\tau) + t_0) - \alpha_-(s(\tau) - t_0) = \beta(s(\tau), t_0) \qquad (4.27) \\
&\overset{(4.20)}{=} 0.
\end{aligned}
$$

Then for all $\tau \in (0, \delta]$, by (4.23) and (4.27), $\beta(\cdot, t_0 - \frac{\tau}{2})$ takes both positive and negative values on $J = [r_3, s(\tau) - \frac{\tau}{2}]$, whilst for all $w_1, w_2 \in J$, arguing as above by (4.22), (4.24) and (4.25) we have $|\alpha_+(w_1 + t_0 - \frac{\tau}{2}) - \alpha_+(w_2 + t_0 - \frac{\tau}{2})| < \varepsilon$ which is what we set out to show. This completes the proof. □

We have now gathered all of the ingredients for the proof of Theorem 4.16

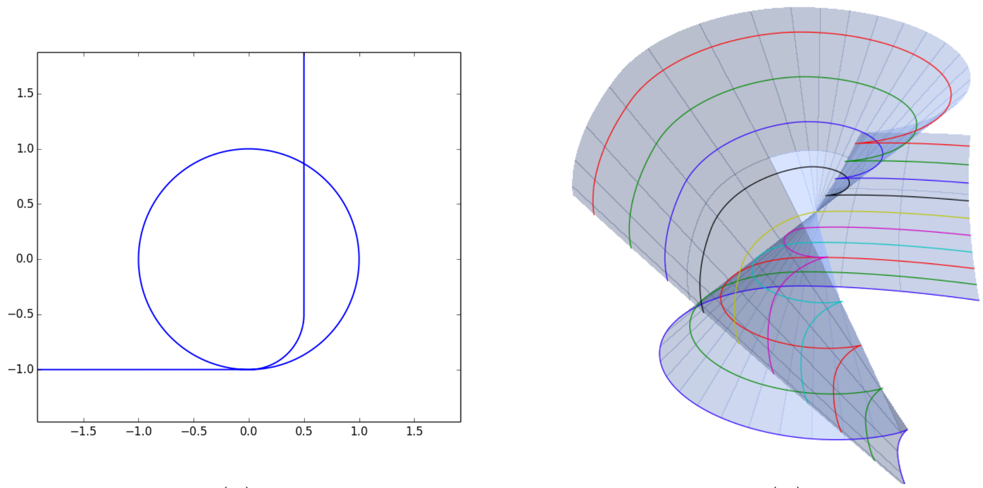
Proof of Theorem 4.16: Letting $\phi(s, t) = (t, \gamma(s, t))$ be an evolution by isothermal gauge for a $C^1 \times C^0$ initial data (C, V), we are supposing that the image of the unit tangent along C contains an arc of length $> \pi$, i.e. there exist $s_1, s_2 \in \mathbb{R}$ for which (4.18) holds. By Lemma 4.17 there then exists a time $t_* \in \mathbb{R}$ such that $\sin \frac{\beta(\cdot,t_*)}{2}$ takes both positive and negative values. By Lemma 4.18 there then exists an open interval I such that for each $t \in I$ either $\mathrm{Im}(\gamma(\cdot,t))$ is not a C^1 immersed curve or $\mathrm{Im}(\gamma(\cdot,t))$ is a C^1 immersed curve but $U(\cdot,t)$ does not admit an extension to a continuous unit tangent vector field along $\gamma(\cdot,t)$. Theorem 4.16 is proved. $\square$

The interval I in Lemma 4.18 may be chosen to be contained in any neighbourhood of the time t_*, but it is not always possible to choose an interval I containing t_*. Indeed, it is possible that $\sin\left(\frac{\beta(\cdot,t_*)}{2}\right)$ takes both positive and negative values on an interval $[s_1, s_2]$ whilst $\gamma([s_1,s_2],t_*)$ is a C^1 immersed curve and $U(\cdot,t_*)$ admits a continuous extension to a unit tangent vector field along $\gamma(\cdot,t_*)$, as the following example illustrates.

Example 4.19 ('Cusp reversal')**.** Consider the C^1 initial curve defined by

$$c(s) = \begin{cases} (s,-1) & \text{for } s \in (-\infty, 0] \\ (\sin s, -\cos s) & \text{for } s \in (0, 2\pi] \\ \left(\frac{1}{2}\sin 2s, \frac{-1}{2}(1+\cos 2s)\right) & \text{for } s \in (2\pi, \frac{9\pi}{4}] \\ \left(\frac{1}{2}, -\frac{1}{2} + s - \frac{9\pi}{4}\right) & \text{for } s \in (\frac{9\pi}{4}, \infty). \end{cases}$$

See Figure 4.6(a). Let $\phi(s,t) = (t, \gamma(s,t))$ be the evolution by isothermal gauge of the curve $C = (0,c)$ with initial velocity $V = (1,0,0)$. We have $\beta(s,\frac{\pi}{2}) < 0$ for $s \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$ and $\beta(s,\frac{\pi}{2}) > 0$ for $s \in (\frac{3\pi}{2}, \frac{3\pi}{2} + \varepsilon]$ for some $\varepsilon > 0$, whilst $\beta(s,\frac{\pi}{2}) = 0$ for $s \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Moreover, $\lim_{s\to\frac{\pi}{2}^-} U(s,\frac{\pi}{2}) = \lim_{s\to\frac{3\pi}{2}^+} U(s,\frac{\pi}{2}) = (0,1)$. Thus $\gamma([\frac{\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon], \frac{\pi}{2})$ is a C^1 immersed curve. See Figure 4.6(b). The numerical plot reveals some interesting geometry at the time $t_* = \frac{\pi}{2}$. We see that at this moment in time a cusp instantaneously reverses the direction of its axis, so that the spatial cross section is C^1 at $\phi(\frac{\pi}{2}, \frac{\pi}{2})$. Although the spatial cross-section is regular at this point, the surface is not, and looks locally like a cone, with a pair of cusps tracing two 'cuts' running down to the vertex.



(a) (b)

Figure 4.6: (a) The initial curve of Example 4.19. (b) The evolution $\phi(s,t)$ of the curve in (a) by isothermal gauge, plotted for values of $s \in [-2, 10]$ and $t \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. The coloured curves are $\{x^0 = \text{constant}\}$ cross sections.

We conclude this section with an example where the set $\gamma([s_1, s_2], t_*)$ is a C^1 immersed curve, whilst $U(\cdot, t_*)$ admits no extension to a continuous unit tangent vector field along $\gamma(\cdot, t_*)$ on $[s_1, s_2]$ (thus $\gamma(\cdot, t_*)$ admits no monotone reparameterisation to a C^1 immersion).

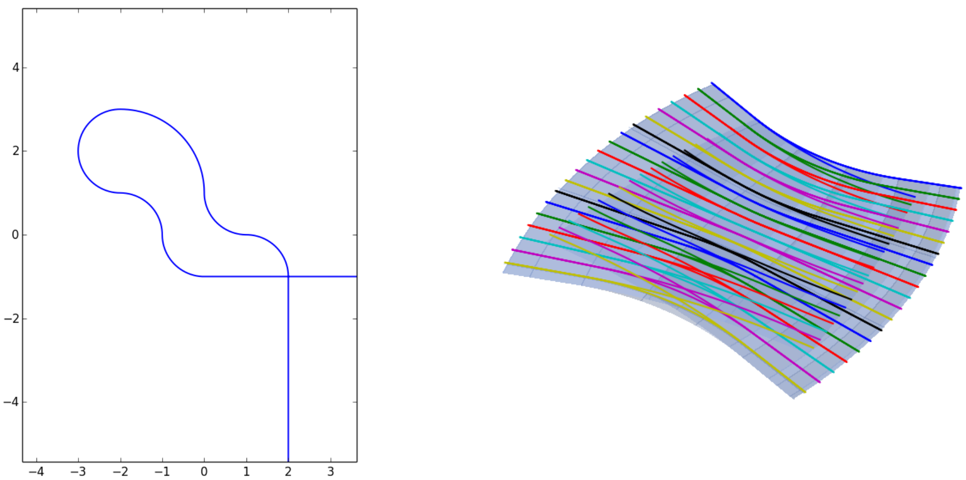
Example 4.20 (Degenerate cusp singularities ('sheeting')). Consider the C^1 initial curve defined by

$$c(s) = \begin{cases} (-s, -1) & \text{for } s \in (-\infty, 0] \\ (-\sin s, -\cos s) & \text{for } s \in (0, \frac{\pi}{2}] \\ \left(-2 + \cos(s - \frac{\pi}{2}), -\sin(s - \frac{\pi}{2})\right) & \text{for } s \in (\frac{\pi}{2}, \pi] \\ (-2 - \sin(s - \pi), 2 - \cos(s - \pi)) & \text{for } s \in (\pi, 2\pi] \\ \left(-2 + 2\sin\frac{s-2\pi}{2}, 1 + 2\cos\frac{s-2\pi}{2}\right) & \text{for } s \in (2\pi, 3\pi] \\ (1 - \cos(s - 3\pi), 1 - \sin(s - 3\pi)) & \text{for } s \in (3\pi, \frac{7\pi}{2}] \\ \left(1 + \sin(s - \frac{7\pi}{2}), -1 + \cos(s - \frac{7\pi}{2})\right) & \text{for } s \in (\frac{7\pi}{2}, 4\pi] \\ (2, -1 - (s - 4\pi)) & \text{for } s \in (4\pi, \infty). \end{cases}$$

See Figure 4.7(a). Let $\phi(s,t) = (t, \gamma(s,t))$ be the evolution of $C(s) = (0, c(s))$ with initial velocity $V = (1, 0, 0)$. It may be seen that the curve $\gamma(s, \frac{3\pi}{2}) = c(s + \frac{3\pi}{2}) + c(s - \frac{3\pi}{2})$ will backtrack and retrace its steps twice, so that the map $s \mapsto U(s, \frac{3\pi}{2}) = {}^{\gamma_s(s,\frac{3\pi}{2})}/_{|\gamma_s(s,\frac{3\pi}{2})|}$ is discontinuous, whilst the image $\gamma([\frac{3\pi}{2}, \frac{5\pi}{2}], \frac{3\pi}{2})$ is a C^1 curve. This phenomenon is illustrated in Figure 4.7(b). In this example, the degenerate behaviour is sandwiched between a pair of ordinary cusps which travel along $t = -s + 2\pi, t > \pi$ and $t = s - \frac{3\pi}{2}, t > \frac{5\pi}{4}$, and the surface Σ is not C^1.

4.5 An example of singularity which is not by collapse for which the limit curve at first singularity is not C^1

Theorem 4.16 shows that if (C, V) is an initial data such that the image of the unit tangent map along C contains an arc of length $> \pi$ then the spatial unit tangent map



(a) (b)

Figure 4.7: (a) The initial curve of Example 4.20. (b) The evolution $\phi(s,t)$ of the curve in (a) by isothermal gauge, plotted for values $s \in [1.4\pi, 2.6\pi], t \in [1.4\pi, 1.6\pi]$. The coloured curves are $\{x^0 = \text{constant}\}$ cross sections

of an evolution by isothermal gauge of (C, V) becomes discontinuous somewhere (i.e. Theorem 4.16). In general, the discontinuity of the spatial unit tangent will occur after the first formation of singularity. Indeed, recall from §1.3.1 that for suitably generic initial data, at the first time of singularity the limit curve will be C^1. A (non-generic) example where the first time of singularity is not a C^1 curve is given by the shrinking circle solution of Example 3.3. In this case the evolution is future inextendible even as a C^0 submanifold beyond the singular time. In the context of spatially non-compact timelike maximal surfaces, however, collapsing singularities are impossible by Lemma 4.1. In this section we will give an example of a smooth proper immersion $C\colon \mathbb{R} \to \mathbb{R}^{1+2}$ and a smooth velocity V along C such that, for the future Cauchy evolution of (C, V) at the first time of singularity, the limit curve is not C^1.

Example 4.21 (Example of a (non-compact) smooth initial data for which singularity formation is not by collapse but the limit curve at the first singularity is not C^1)**.** Let $q\colon [0, L] \to \mathbb{R}^2$ be a smooth embedding such that $|\dot{q}(s)|^2 = 1$ for all $s \in [0, L]$ (so that the length of q is L), such that $q(0) = (0, -1)$, $q(L) = (1, 0)$, $\dot{q}(0) = (1, 0)$,

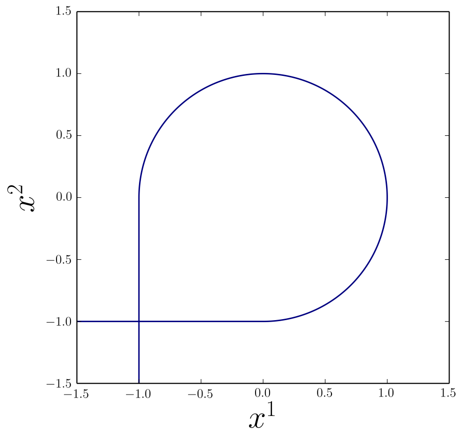
$\dot{q}(L) = (0,1)$ and $\frac{d^k q}{ds^k}(0) = \frac{d^k q}{ds^k}(L) = 0$ for all $k \geq 2$, and such that $\ddot{q}^2(s) > 0$ for all $s \in (0, L)$. Let $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ denote anti-clockwise rotation of the plane by $\frac{\pi}{2}$ radians and define a smooth immersion $C \colon \mathbb{R} \to \mathbb{R}^{1+2}$ by $C(s) = (0, c(s))$ where

$$c(s) = \begin{cases} (s, -1) & \text{for } s \in (-\infty, 0) \\ q(s) & \text{for } s \in [0, L) \\ Rq(s-L) & \text{for } s \in [L, 2L) \\ R^2 q(s-2L) & \text{for } s \in [2L, 3L) \\ (-1, -(s-3L)) & \text{for } s \in [3L, \infty). \end{cases}$$

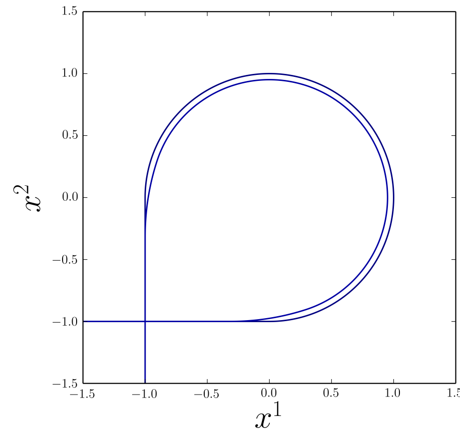
See Figure 4.8(i) for a C^1 approximation of such a curve. Note that we have $c'(s) = (\cos\vartheta(s), \sin\vartheta(s))$ where $\vartheta(s) = 0$ for $s \in (-\infty, 0]$, $\vartheta(s) = \frac{3\pi}{2}$ for $s \in [3L, \infty)$ and $\vartheta(s)$ is strictly increasing for $s \in (0, 3L)$, and where $\vartheta(s+L) = \vartheta(s) + \frac{\pi}{2}$ for $s \in [0, 2L]$. Let $V(s) = (1, 0, 0)$ be the initial velocity along C and let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be the evolution of (C, V) by isothermal gauge. Then ϕ is a smooth map of the form $\phi(s,t) = (t, \gamma(s,t))$ where $\gamma(s,t) = \frac{1}{2}\left(c(s+t) + c(s-t)\right)$. We claim that ϕ is an immersion for $|t| < L$ and $\operatorname{Im}(\gamma(\cdot, L))$ is not a C^1 immersed curve (so the first time of singularity is $t = L$ and the limit curve is not C^1). Indeed, by construction we have that $|\vartheta(s+t) - \vartheta(s-t)| < \pi$ for all $s \in \mathbb{R}$ provided $|t| < L$ and $|\vartheta(s+L) - \vartheta(s-L)| = \pi$ iff $s \in [L, 2L]$. So ϕ is an immersion for $|t| < L$ and $\gamma(\cdot, L)$ is an immersion at s iff $s \notin [L, 2L]$. This shows that $\operatorname{Im}(\gamma(\cdot, L)) = \gamma((-\infty, L], L) \cup \gamma([2L, \infty), L)$ and moreover we may compute that

$$\lim_{s \uparrow L} \frac{\gamma_s(s,L)}{|\gamma_s(s,L)|} = (0,1); \qquad \lim_{s \downarrow 2L} \frac{\gamma_s(s,L)}{|\gamma_s(s,L)|} = (-1,0).$$

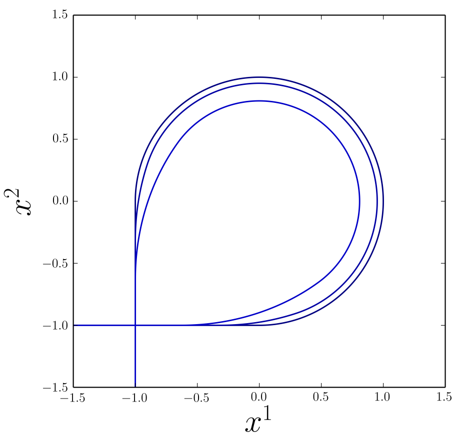
Thus $\operatorname{Im}(\gamma(\cdot, L))$ is not a C^1 immersed curve (in fact it contains a right-angled corner) as claimed. See Figure 4.8.



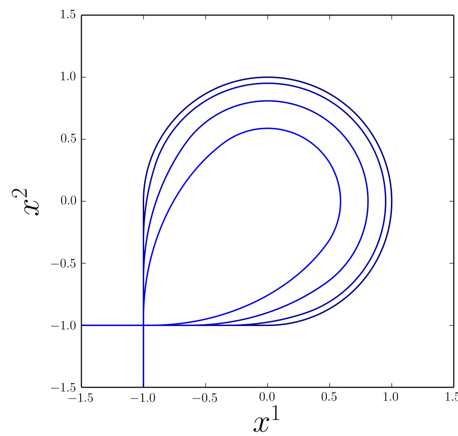
(i)



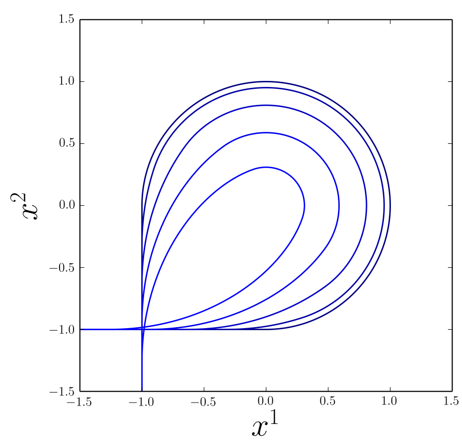
(ii)



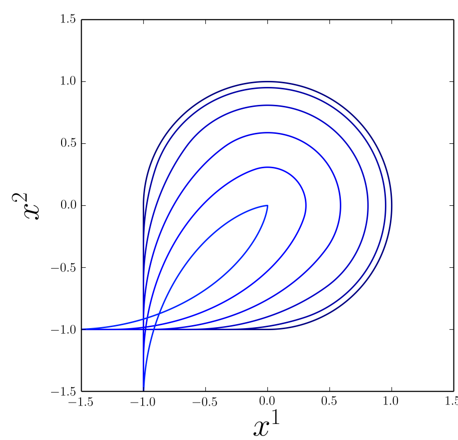
(iii)



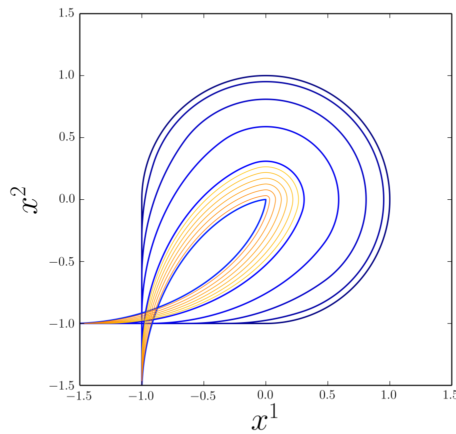
(iv)



(v)



(vi)



(vii)

Figure 4.8: The Cauchy evolution of Example 3.3 in which the limit curve at the first time of singularity is not C^1. The $\{x^0 = t\}$ cross sections of the timelike maximal surface are plotted in blue for (i) $t = 0$, (ii) $t = .1\pi$, (iii) $t = .2\pi$, (iv) $t = .3\pi$, (v) $t = .4\pi$ and (vi) $t = .5\pi$. In (vii) we resolve the motion in a neighbourhood of the singularity by plotting in orange the $\{x^0 = t\}$ cross sections as t varies in 6 evenly spaced intervals from $t = .415\pi$ to $t = .49\pi$.

Chapter 5

An initial-boundary value problem

Figure 5.1: Waves reflected off a boundary at Christ Church Meadow, May 2019.

5.0 A piece of notation

For the rest of this thesis we will adopt the shorthand

$$\langle\!\langle V, W\rangle\!\rangle = -V^0W^0 + v^1w^1 + v^2w^2 = -V^0W^0 + \langle v, w\rangle$$
$$\|V\|^2 = \langle\!\langle V, V\rangle\!\rangle = -(V^0)^2 + |v|^2$$

for the Minkowskian inner product between two vectors $V = (V^0, v) \in \mathbb{R}^{1+2}$ and $W = (W^0, w) \in \mathbb{R}^{1+2}$.

5.1 Statement of the IBVP for a timelike maximal surface with a single prescribed timelike boundary curve

Let $\Gamma\colon [0,\infty) \to \mathbb{R}^{1+2}$ be a C^2 proper future-directed timelike immersion, let $C\colon [0,\infty) \to \mathbb{R}^{1+2}$ be a C^2 proper immersion of the form $C(s) = (0, c(s))$ and such that

$$C(0) = \Gamma(0) \tag{5.1}$$

and let V be a C^1 future-directed timelike vector field along C such that

$$\dot{\Gamma}(0) \in \operatorname{span}\{C'(0), V(0)\}. \tag{5.2}$$

We refer to such (C, V, Γ) as an initial-boundary data. Given such an initial-boundary data (C, V, Γ), the initial-boundary value problem (IBVP) is to find a C^2 proper timelike maximal immersion $\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ such that $t \mapsto \phi(0,t)$ is a monotone reparameterisation of Γ and $s \mapsto \phi(s,0)$ is a monotone reparameterisation of C with $\mathrm{Im}(\phi)$ contained in the future of $\mathrm{Im}(C)$ and with V tangent to $\mathrm{Im}(\phi)$ along C.

In order to find a C^2 solution to the IBVP, there is a natural C^2 compatibility condition that the data (C, V, Γ) should satisfy. Let us quickly derive this condition, which effectively states that the mean curvature should vanish 'at the corner'. Suppose that $\phi \colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ is a C^2 solution to the IBVP. Reparameterising as necessary, we may assume that $\phi \colon [0,a) \times [0,b) \to \mathbb{R}^{1+2}$ is such that $l \mapsto \phi(l,0)$ is a reparametrisation of C by arclength whilst $\tau \mapsto \phi(0,\tau)$ is a reparametrisation of Γ by arclength. Denote by $l \mapsto N(l)$ a unit vector field along C which is normal to the plane $\mathrm{span}\{\frac{dC}{dl}(l), V(l)\}$ (so that N is a unit normal to the surface $\Sigma = \mathrm{Im}(\phi)$ along C). The mean curvature scalar is then computed at the point $(0,0)$ as

$$
\begin{aligned}
h(0,0) = \frac{1}{-1 - \langle\!\langle \phi_l(0,0), \phi_\tau(0,0) \rangle\!\rangle^2} \Bigg(& - \langle\!\langle \phi_{ll}(0,0), N(0) \rangle\!\rangle \\
& + 2 \langle\!\langle \phi_l(0,0), \phi_\tau(0,0) \rangle\!\rangle \left\langle\!\!\left\langle \phi_\tau(0,0), \frac{dN}{dl}(0) \right\rangle\!\!\right\rangle + \langle\!\langle \phi_{\tau\tau}(0,0), N(0) \rangle\!\rangle \Bigg) = 0
\end{aligned}
$$

and we arrive at the condition

$$
-\left\langle\!\!\left\langle \frac{d^2C}{dl^2}(0), N(0) \right\rangle\!\!\right\rangle + 2 \left\langle\!\!\left\langle \frac{dC}{dl}(0), \frac{d\Gamma}{d\tau}(0) \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \frac{d\Gamma}{d\tau}(0), \frac{dN}{dl}(0) \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{d^2\Gamma}{d\tau^2}(0), N(0) \right\rangle\!\!\right\rangle = 0 \tag{5.3}
$$

for the data (C, V, Γ) where l is the arclength parameter along C and τ is the arclength parameter along Γ. Condition (5.3) is the statement that the mean curvature vanishes at the corner. In the next section, we will arrive at an additional (less natural) C^2 compatibility condition.

5.2 A choice of conformal structure

In the last section we introduced an IBVP for a single timelike boundary curve. We would now like to study this IBVP by conformal methods. One of the interesting aspects of Lorentzian geometry is the rich variety of possible conformal structures. Whilst any Riemannian surface diffeomorphic to $\mathbb{R}^2$ is smoothly conformally equivalent to either the plane or the unit disc, there exist infinitely many smooth Lorentzian surfaces diffeomorphic to $\mathbb{R}^2$ no two of which are C^0 conformally equivalent. Moreover, C^k conformal equivalence between Lorentz surfaces does not imply C^{k+1} conformal equivalence. In fact, for every $k \in \mathbb{N} \cup \{0\}$ there exist infinitely many smooth and C^k conformally equivalent Lorentzian surfaces diffeomorphic to $\mathbb{R}^2$, no two of which are C^{k+1} conformally equivalent [70, p.45 Theorem 2']. Given such a complex state of affairs, we will restrict our attention in this thesis and seek C^2 solutions to the IBVP with a simple C^2 conformal structure. It will transpire, however, that one aspect of this conformal structure is justified a postiori.

Let (s, t) be coordinates on $\mathbb{R}^{1+1}$ so that the Minkowski metric is $ds^2 - dt^2$. For $\mu \in (-1, 1)$ and $T_* \in (0, \infty]$ we define the conformal domain

$$\Omega_\mu^{T_*} = \left\{ (s,t) \colon 0 \leq t, \mu t \leq s, \frac{t-s}{1-\mu} < T_* \right\} \subseteq \mathbb{R}^{1+1}, \tag{5.4}$$

see Figure 5.2. By a rescaling of $\mathbb{R}^{1+1}$ it is clear that for any $T_1, T_2 \in (0, \infty)$ the domains $\Omega_\mu^{T_1}$ and $\Omega_\mu^{T_2}$ are smoothly conformally equivalent. On the other hand, it may be shown that for $\mu_1 \neq \mu_2$ the domains $\Omega_{\mu_1}^{T_*}$ and $\Omega_{\mu_2}^{T_*}$ are C^1 conformally distinct, whilst the domains Ω_μ^∞ and Ω_μ^1 are C^0 conformally distinct, see [46]. One may think of Ω_μ^∞ and Ω_μ^1 as describing the two simplest possible structures of null infinity.

We will seek C^2 solutions to the IBVP with the C^2 conformal structure $\Omega_\mu^{T_*}$ for some $\mu \in (-1, 1)$, $T_* \in (0, \infty]$. It turns out that this choice of C^2 conformal structure

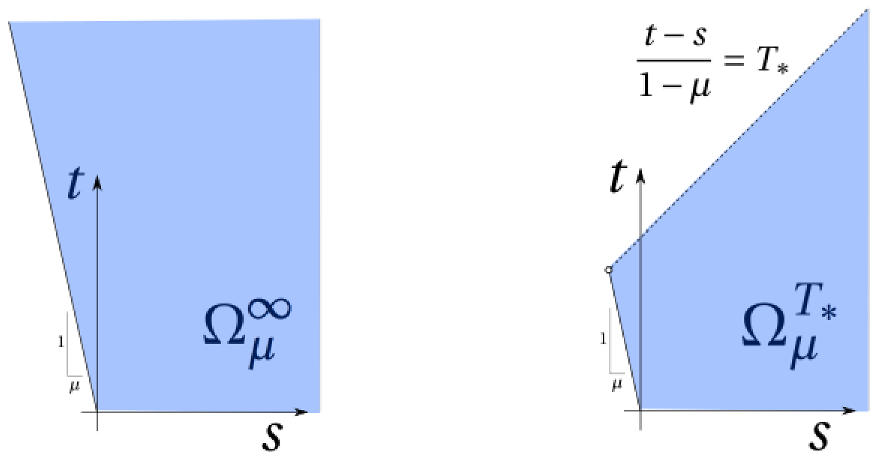


Figure 5.2: Conformal domain for solution to the initial-boundary value problem: the cases $T_* = \infty$ and $T_* < \infty$.

imposes C^2 compatibility conditions on the initial data (C, V, Γ) in addition to (5.3) (specifically, it imposes local constraints 'at the corner') which we will now proceed to derive. Suppose that

$$\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$$

is a C^2 proper timelike immersion with vanishing mean curvature which is conformal with respect to the metric $ds^2 - dt^2$ on $\Omega_\mu^{T_*}$ such that $t \mapsto \phi(\mu t, t)$ is future-directed timelike and $\mathrm{Im}(\phi(\cdot, 0)) \subseteq \{x^0 = 0\}$. So ϕ satisfies

$$\|\phi_t \pm \phi_s\|^2 = 0 \tag{5.5}$$

$$\phi_{tt} - \phi_{ss} = 0. \tag{5.6}$$

We may assume $\phi(0,0) = (0,0,0)$ without loss of generality. Writing $\phi(s,0) = C(s) = (0, c(s))$ and $\phi_t(s,0) = V(s)$ and writing

$$\phi(0,t) = \Gamma(z(t))$$

where $\Gamma\colon [0,\infty) \to \mathbb{R}^{1+2}$ is taken to be parameterised as

$$\Gamma(x^0) = (x^0, \sigma(x^0))$$

and where

$$z\colon [0,T_*) \to [0,\infty)$$

is some C^2 strictly increasing and surjective function, one may compute d'Alembert's formula

$$\phi(s,t) = \begin{cases} \frac{1}{2}\left(C(s+t) + C(s-t) + \int_{s-t}^{s+t} V(\zeta)d\zeta\right) & \text{for } s \geq t \\ \frac{1}{2}\left(C(s+t) - C\left(\frac{1+\mu}{1-\mu}(t-s)\right) + \int_{\frac{1+\mu}{1-\mu}(t-s)}^{t+s} V(\zeta)d\zeta\right) & \\ \quad +\Gamma\left(z\left(\frac{t-s}{1-\mu}\right)\right) & \text{for } t > s. \end{cases} \tag{5.7}$$

Since $\langle\!\langle \phi_s(\mu t,t), \phi_t(\mu t,t)\rangle\!\rangle = 0$ for all $t \in [0,T_*)$ it may be computed from (5.7) that $z\colon [0,T_*) \to [0,\infty)$ satisfies

$$\begin{aligned} \dot{z}(t) &= (1+\mu)\frac{\langle\!\langle \frac{d\Gamma}{dx^0}(z(t)), A_+\left((1+\mu)t\right)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(z(t))\|^2}, \\ z(0) &= 0 \end{aligned} \tag{5.8}$$

where $A_+(s) = V(s) + C'(s)$ as usual.

Since ϕ is C^1 we have $\lim_{t\downarrow 0} \frac{d}{dt}\phi(\mu t,t) = \lim_{s\downarrow 0}\left(\phi_t(s,0) + \mu\phi_s(s,0)\right)$ which implies

$$\frac{d\Gamma}{dx^0}(0)\dot{z}(0) = V(0) + \mu C'(0). \tag{5.9}$$

Note that the equation (5.9) expresses the parameter $\mu \in (-1,1)$ in terms of (C,V,Γ).

By rescaling $\phi(s,t) \mapsto \phi(\lambda s, \lambda t)$ and redefining T_* if necessary we may take that

$$\frac{d\Gamma}{dx^0}(0) = V(0) + \mu C'(0), \tag{5.10}$$

or in other words, λ is chosen so that the solution $z\colon [0, T_*) \to [0, \infty)$ of (5.8) satisfies

$$\dot{z}(0) = 0. \tag{5.11}$$

Since ϕ is C^2 and satisfies (5.6) we have

$$\lim_{t \downarrow 0} \frac{d^2}{dt^2}\phi(\mu t, t) = \lim_{s \downarrow 0} \Big((1+\mu^2)\phi_{ss}(s,0) + 2\mu\phi_{st}(s,0) \Big)$$

which implies by (5.7) and (5.11) that

$$\frac{d^2\Gamma}{(dx^0)^2}(0) + \frac{d\Gamma}{dx^0}(0)\ddot{z}(0) = (1+\mu^2)C''(0) + 2\mu V'(0). \tag{5.12}$$

The x^0 component of (5.12) reads

$$\ddot{z}(0) = 0 \tag{5.13}$$

and so the x^1–x^2 components of (5.12) read

$$\frac{d^2\sigma}{(dx^0)^2}(0) = (1+\mu^2)c''(0) + 2\mu v'(0). \tag{5.14}$$

We will now show that (5.14) implies (5.13). Indeed, supposing that (5.14) holds, we compute

$$
\begin{aligned}
\ddot{z}(0) &= (1+\mu)\Bigg((1+\mu)\frac{\langle\!\langle \frac{d\Gamma}{dx^0}(0), A'_+(0)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(0)\|^2} + \frac{\langle\!\langle \frac{d^2\Gamma}{(dx^0)^2}(0), A_+(0)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(0)\|^2}\dot{z}(0) \\
&\quad - 2\frac{\langle\!\langle \frac{d\Gamma}{dx^0}(0), A_+(0)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(0)\|^4}\langle\!\langle \frac{d\Gamma}{dx^0}(0), \frac{d^2\Gamma}{(dx^0)^2}(0)\rangle\!\rangle\dot{z}(0)\Bigg) \\
&= \frac{1}{(1-\mu)(-1+|v(0)|^2)}\Bigg((1+\mu)\left\langle c''(0)+v'(0), \mu c'(0)+v(0)\right\rangle \\
&\quad + \left\langle c'(0)+v(0), \frac{d^2\sigma}{(dx^0)^2}(0)\right\rangle - \frac{2}{1+\mu}\left\langle \mu c'(0)+v(0), \frac{d^2\sigma}{(dx^0)^2}(0)\right\rangle\Bigg) \\
&= \frac{1}{-1+|v(0)|^2}\Bigg(\frac{1+\mu}{1-\mu}\left\langle c''(0)+v'(0), \mu c'(0)+v(0)\right\rangle \\
&\quad + \frac{1}{1+\mu}\left\langle c'(0)-v(0), \frac{d^2\sigma}{(dx^0)^2}(0)\right\rangle\Bigg) \\
&= \frac{1}{-1+|v(0)|^2}\Bigg(\frac{1+\mu}{1-\mu}\left\langle c''(0)+v'(0), \mu c'(0)+v(0)\right\rangle \\
&\quad + \frac{1}{1+\mu}\left\langle c'(0)-v(0), (1+\mu^2)c''(0)+2\mu v'(0)\right\rangle\Bigg) \\
&= \frac{1}{-1+|v(0)|^2}\frac{1+3\mu^2}{1-\mu^2}\Big(\left\langle c''(0), c'(0)\right\rangle + \left\langle v'(0), v(0)\right\rangle \\
&\quad + \left\langle c''(0), v(0)\right\rangle + \left\langle c'(0), v'(0)\right\rangle\Big) \\
&= 0
\end{aligned}
$$

where we appealed to the formula (5.10) together with the identities $\langle c'(s), v(s)\rangle = 0$ and $|c'(s)|^2 + |v(s)|^2 = 1$. Thus we have shown that (5.12) is equivalent to (5.14).

For the purpose of clarity, let us express the right hand side of (5.14) in terms of arclength parameter l along C. Let

$$
l(s) = \int_0^s \sqrt{|c'(\tilde{s})|^2}d\tilde{s} = \int_0^s \sqrt{1-|v(\tilde{s})|^2}d\tilde{s}
$$

denote the arclength parameter along C. Then we have

$$\begin{aligned} v'(0) &= \sqrt{1-|v(0)|^2}\frac{dv}{dl}(0) \\ c''(0) &= (1-|v(0)|^2)\frac{d^2c}{dl^2}(0) - \left\langle v(0), \frac{dv}{dl}(0)\right\rangle \frac{dc}{dl}(0) \end{aligned}$$

so (5.14) reads

$$\begin{aligned} \frac{d^2\sigma}{(dx^0)^2}(0) = (1+\mu^2)&\left((1-|v(0)|^2)\frac{d^2c}{dl^2}(0) - \left\langle v(0), \frac{dv}{dl}(0)\right\rangle \frac{dc}{dl}(0)\right) \\ &+ 2\mu\sqrt{1-|v(0)|^2}\frac{dv}{dl}(0). \end{aligned} \tag{5.15}$$

We now pose the following IBVP.

IBVP 5.1. Given an initial-boundary data (C, V, Γ) which satisfies the C^2 compatibility condition (5.15), find a C^2 proper timelike maximal immersion $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ for some $\mu \in (-1,1)$ and $T_* \in (0,\infty]$ where $\Omega_\mu^{T_*}$ is as in (5.4), which is a conformal map with respect to the Minkowski metric $ds^2 - dt^2$ on $\Omega_\mu^{T_*}$, such that $s \mapsto \phi(s,0)$ is a monotone reparameterisation of C, $t \mapsto \phi(\mu t, t)$ is a monotone reparameterisation of Γ, $\mathrm{Im}(\phi)$ is contained in the future of $\mathrm{Im}(C)$ and V is tangent to $\mathrm{Im}(\phi)$ along C.

Remark 5.2 (The simplest case of (5.15))**.** The C^2 compatibility condition (5.15) is not particularly intuitive, so let us point out a simple (and trivial) case for which (5.15) holds. Suppose that the boundary curve Γ has zero curvature at 0 (so $\frac{d^2\Gamma}{d\tau^2}(0) = 0$ where τ is the arclength of Γ), the initial curve C has zero curvature at 0 (so $\frac{d^2C}{dl^2}(0) = 0$ where l is the arclength of C), and the normal vector $N(l)$ to the timelike plane $\mathrm{span}\{\frac{dC}{dl}(l), V(l)\}$ is parallel along C at 0 (so $\frac{dN}{dl}(0) = 0$). Then it may be seen to follow that $\frac{d^2\sigma}{(dx^0)^2}(0) = \frac{d^2c}{dl^2}(0) = \frac{dv}{dl}(0) = 0$ so (5.15) holds trivially in this case.

5.3 Evolution by isothermal gauge

Let (C, V, Γ) be an initial-boundary data which satisfies the C^2 compatibility condition (5.15). We now look to solve IBVP 5.1 by constructing the desired conformal timelike maximal immersion, and we will start by parameterising the data accordingly. We may take V to be of the form $V(s) = (1, v(s))$ where

$$\langle c'(s), v(s) \rangle = 0 \tag{5.16}$$

and we may then uniquely monotonically reparametrise $C\colon [0, \infty) \to \mathbb{R}^{1+2}$, $C(s) = (0, c(s))$ so that

$$|c'(s)|^2 + |v(s)|^2 = 1. \tag{5.17}$$

Note that $\|V(s) \pm C'(s)\|^2 = 0$ so the vectors

$$A_\pm(s) = V(s) \pm C'(s) = (1, v(s) \pm c'(s)) = (1, a_\pm(s))$$

are null vectors which span the tangent plane to the prospective timelike maximal surface at $C(s)$. It is clear that we still have a proper immersion of the form $C\colon [0, \infty) \to \mathbb{R}^{1+2}$ after reparametrisation. We may take $\Gamma(0) = C(0) = (0, 0, 0)$ without loss of generality and, as Γ is proper and timelike, we may take it to be parameterised as

$$\Gamma(x^0) = (x^0, \sigma(x^0)). \tag{5.18}$$

By (5.2) there is a $\mu \in (-1, 1)$ such that

$$\frac{d\Gamma}{dx^0}(0) = V(0) + \mu C'(0). \tag{5.19}$$

Recalling (5.8) from §5.2, we now turn to analyse the following ordinary differential equation

$$\begin{aligned} \dot{z}(t) &= (1+\mu)\frac{\langle\!\langle \frac{d\Gamma}{dx^0}(z(t)), A_+((1+\mu)t)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(z(t))\|^2}, \\ z(0) &= 0. \end{aligned} \tag{5.20}$$

Since Γ is C^2 and timelike and A_+ is C^1, the right hand side of (5.20) is a C^1 function of z and t. So by the Picard theorem (see e.g. [20, Chap. 1]) there is a unique C^2 solution $z\colon [0, T) \to [0, \infty)$ to (5.20) for some $T > 0$. Let $[0, T_*)$ denote the maximal interval of existence for (5.20) and $z\colon [0, T_*) \to [0, \infty)$ denote the unique inextendible solution to (5.20). We claim that z is strictly increasing and surjective (i.e. a diffeomorphism). Indeed, since $|\frac{d\sigma}{dx^0}| < 1$ we have

$$\begin{aligned} \dot{z}(t) &= (1+\mu)\frac{\langle\!\langle \frac{d\Gamma}{dx^0}(z), A_+((1+\mu)t)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(z)\|^2} \\ &= (1+\mu)\frac{1 - \langle \frac{d\sigma}{dx^0}(z), a_+((1+\mu)t)\rangle}{1 - |\frac{d\sigma}{dx^0}(z)|^2} \\ &\geq (1+\mu)\frac{1 - |\frac{d\sigma}{dx^0}(z)|}{1 - |\frac{d\sigma}{dx^0}(z)|^2} = \frac{(1+\mu)}{1 + |\frac{d\sigma}{dx^0}(z)|} \geq \frac{1+\mu}{2} \end{aligned} \tag{5.21}$$

so z is strictly increasing. If $T_* < \infty$ we know from the Picard theorem that $\lim_{t\uparrow T_*} z(t) = \infty$ whilst if $T_* = \infty$ then (5.21) implies that $\lim_{t\uparrow\infty} z(t) = \infty$. In either case we have shown $z\colon [0, T_*) \to [0, \infty)$ is strictly increasing and surjective as claimed.

Remark 5.3. Suppose that $\Gamma\colon [0, \infty) \to \mathbb{R}^{1+2}$, $\Gamma(x^0) = (x^0, \sigma(x^0))$ is a uniformly timelike curve, so that there exists a $b \in [0,1)$ such that $|\frac{d\sigma}{dx^0}(x^0)| \leq b$ for all $x^0 \in [0,\infty)$. Then it follows from (5.20) that

$$
\begin{aligned}
\dot{z}(t) &= (1+\mu)\frac{\langle\!\langle \frac{d\Gamma}{dx^0}(z), A_+((1+\mu)t)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(z)\|^2} \\
&= (1+\mu)\frac{1-\langle \frac{d\sigma}{dx^0}(z(t)), a_+((1+\mu)t)\rangle}{1-|\frac{d\sigma}{dx^0}(z(t))|^2} \leq (1+\mu)\frac{1+|\frac{d\sigma}{dx^0}(z(t))|}{1-|\frac{d\sigma}{dx^0}(z(t))|^2} \\
&= \frac{(1+\mu)}{1-|\frac{d\sigma}{dx^0}(z(t))|} \leq \frac{1+\mu}{1-b},
\end{aligned}
$$

which is a uniform upper bound. Thus if Γ is uniformly timelike then $T_* = \infty$.

Next, we recall from (5.4) the conformal domain

$$
\Omega_\mu^{T_*} = \left\{(s,t)\colon 0 \leq t, \mu t \leq s, \frac{t-s}{1-\mu} < T_*\right\} \subseteq \mathbb{R}^{1+1}
$$

(see Figure 5.2) and we define

$$
\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}
$$

to be the unique solution to the initial-boundary value problem

$$
\phi_{tt} - \phi_{ss} = 0 \tag{5.22}
$$

$$
\phi(s,0) = C(s) \tag{5.23}
$$

$$
\phi_t(s,0) = V(s) \tag{5.24}
$$

$$
\phi(\mu t, t) = \Gamma(z(t)), \tag{5.25}
$$

which is given explicitly by the d'Alembert formula (5.7). As in the boundary-less case, we adopt the following definition.

Definition 5.4. Let (C, V, Γ) be a $C^2 \times C^1 \times C^2$ initial boundary data satisfying the C^2 compatibility condition (5.15) and take the data (C, V, Γ) to be monotone reparametrised according to (5.16)–(5.18). Let $\mu \in (-1, 1)$ denote the constant in (5.19), let $z\colon [0, T_*) \to [0, \infty)$ denote the inextendible solution to (5.20), let $\Omega_\mu^{T_*}$ be the conformal domain (5.4) and let $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ satisfy the initial-boundary problem (5.22)–(5.25). Then we call ϕ the evolution of (C, V, Γ) by isothermal gauge.

The following result shows that if the evolution by isothermal gauge is an immersion, then it gives a (global) solution to IBVP 5.1.

Proposition 5.5. *Let (C, V, Γ) be a $C^2 \times C^1 \times C^2$ initial-boundary data satisfying the C^2 compatibility condition (5.15) and let $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ be the evolution of (C, V, Γ) by isothermal gauge as in Definition 5.4. Then ϕ satisfies the following properties:*

1. *ϕ is C^2.*

2. *ϕ is proper.*

3. *ϕ satisfies $\phi_t^0 \geq \frac{1}{2}$ as well as the isothermal conditions $\|\phi_t \pm \phi_s\|^2 = 0$ on $\Omega_\mu^{T_*}$ and ϕ is a timelike maximal immersion on $\Omega_\mu^{T_*} \setminus \mathcal{K}$ where*

$$\mathcal{K} = \Big\{(s,t) \in \Omega_\mu^{T_*} \colon \operatorname{rank}(d\phi) \neq 2\Big\} = \Big\{(s,t) \in \Omega_\mu^{T_*} \colon \|\phi_s(s,t)\|^2 = 0\Big\}. \tag{5.26}$$

Proof. We start by showing that ϕ is C^2. Note that from the d'Alembert formula (5.7) it follows immediately that ϕ is C^2 for $s > t$ and for $s < t$ so to show that ϕ is C^k (k=0,1,2) it suffices to show that ϕ is C^k along the line $s = t$. Moreover, it may be checked from (5.7) that ϕ will be C^k along the line $s = t$ provided it is C^k at the point $(s,t) = (0,0)$. So ϕ is C^0 provided $\Gamma(0) = C(0)$ which is precisely the first compatibility condition (5.1). Now let us show that ϕ is C^1. From (5.7) one may check that ϕ is C^1 provided $\lim_{t\downarrow 0} \frac{d}{dt}\Big(\phi(\mu t, t)\Big) = \lim_{s\downarrow 0} \left(\mu\phi_s(s,0) + \phi_t(s,0)\right)$ which is equivalent to

$$\frac{d\Gamma}{dx^0}(0)\dot{z}(0) = \mu C'(0) + V(0). \tag{5.27}$$

Recalling that we chose μ (by invoking the second compatibility condition (5.2)) to satisfy (5.19), we compute

$$\dot{z}(0) = (1+\mu)\frac{\langle\!\langle \mu C'(0) + V(0), C'(0) + V(0)\rangle\!\rangle}{\|\mu C'(0) + V(0)\|^2} = (1+\mu)\frac{1-\mu}{1-\mu^2} = 1 \tag{5.28}$$

and (5.27) follows from (5.19), so ϕ is C^1. Now let us show that ϕ is C^2. From (5.7) one may check that ϕ is C^2 provided

$$\lim_{t\downarrow 0}\frac{d^2}{dt^2}\phi(\mu t, t) = \lim_{s\downarrow 0}\Big((1+\mu^2)\phi_{ss}(s,0) + 2\mu\phi_{st}(s,0)\Big)$$

which is equivalent by (5.28) to

$$\frac{d\Gamma}{dx^0}(0)\ddot{z}(0) + \frac{d^2\Gamma}{(dx^0)^2}(0) = (1+\mu^2)C''(0) + 2\mu V'(0) \tag{5.29}$$

and by the computations from (5.13) to (5.15) we recall that (5.29) is equivalent to the third compatibility condition (5.15). So ϕ is C^2.

We will now show that ϕ is proper. To show that ϕ is proper, we take a sequence $x_k = (s_k, t_k) \in \Omega_0^{T_*}$ which exits every compact set, so that either $\limsup s_k = \infty$ or $\limsup \frac{t_k - s_k}{1-\mu} = T_*$, and we claim that $\limsup |\phi(x_k)| = \infty$. Passing to a subsequence we may assume either $t_k \leq s_k$ for all k or $t_k \geq s_k$ for all k. We treat first the case that $t_k \leq s_k$ for all k. From (5.7) we have $\phi^0(s_k, t_k) = t_k$, so if $\limsup t_k = \infty$ we are done. Thus we may take that $t_k \leq C$ and $\limsup s_k = \infty$. But then writing $\phi(s,t) = (t, \gamma(s,t))$, since $|\gamma_t(s,t)| \leq 1$ it follows $|\gamma(s_k, t_k) - \gamma(s_k, 0)| \leq \int_0^{t_k} |\gamma_t(s_k, \tau)| d\tau \leq C$ and since $\limsup |\gamma(s_k, 0)| = \infty$ (because C is proper) we deduce that $\limsup|\gamma(s_k,t_k)| = \infty$ so we are also done. Next we treat the case that $t_k \geq s_k$ for all k. From (5.7) we have $\phi^0(s_k,t_k) = \frac{s_k-\mu t_k}{1-\mu} + z\left(\frac{t_k-s_k}{1-\mu}\right)$ where we recall that $z\colon [0,T_*) \to [0,\infty)$ is strictly increasing and surjective. Since $s_k - \mu t_k \geq 0$ it follows that if $\limsup \frac{t_k-s_k}{1-\mu} = T_*$ then $\limsup \phi^0(s_k,t_k) \geq \limsup z\left(\frac{t_k-s_k}{1-\mu}\right) = \infty$, whilst if $\frac{t_k-s_k}{1-\mu} \leq C < T_*$ then $\limsup \phi^0(s_k,t_k) \geq \limsup \frac{s_k-\mu t_k}{1-\mu} \geq \limsup s_k - \mu C = \infty$. We have now treated all possible cases and shown $\limsup|\phi(x_k)| = \infty$ as claimed. So ϕ is proper.

Next we will show that ϕ satisfies $\phi^0_t \geq \frac{1}{2}$ as well as the isothermal conditions $\|\phi_t \pm \phi_s\|^2 = 0$. From the d'Alembert formula (5.7) we have $\phi^0_t(s,t) = 1$ for $s \geq t$ and from (5.7) and (5.21) we have

$$\phi^0_t(s,t) = \frac{-\mu}{1-\mu} + \frac{1}{1-\mu}\dot{z}\left(\frac{t-s}{1-\mu}\right) \geq \frac{-\mu}{1-\mu} + \frac{1}{1-\mu}\frac{1+\mu}{2} = \frac{1}{2} \tag{5.30}$$

for $t > s$, so $\phi^0_t \geq \frac{1}{2}$. That ϕ satisfies the isothermal conditions $\|\phi_t \pm \phi_s\|^2 = 0$ could be checked directly from (5.7), but let us give a more intuitive proof. Define

$$Z_\pm(s,t) = \phi_t(s,t) \pm \phi_s(s,t)$$

so that what we want to show is $\|Z_\pm(s,t)\|^2 = 0$ for all $(s,t) \in \Omega^{T_*}_\mu$ (i.e. the vectors $Z_\pm$ are null). From (5.16) and (5.17) we have

$$\|Z_\pm(s,0)\|^2 = 0 \tag{5.31}$$

for all $s \in [0,\infty)$ and since the boundary parametrization z satisfies (5.20) one may compute from (5.7) that the condition

$$\langle\langle \phi_s(\mu t,t), \phi_t(\mu t,t)\rangle\rangle = 0 \tag{5.32}$$

is satisfied for all $t \in [0, T_*)$. Since ϕ satisfies the wave equation (5.22), it follows $(\partial_t \mp \partial_s) Z_\pm = 0$ so it follows $\|Z_+(s,t)\|^2$ is constant along lines $s+t =$ constant whilst $\|Z_-(s,t)\|^2$ is constant along lines $s - t =$ constant. From (5.31) then we deduce $\|Z_+(s,t)\|^2 = 0$ for all $(s,t) \in \Omega_\mu^{T_*}$ and $\|Z_-(s,t)\|^2 = 0$ for all $(s,t) \in \Omega_\mu^{T_*} \cap \{s \geq t\}$. But then, by the identity $\|Z_+\|^2 - \|Z_-\|^2 = 4\langle\!\langle \phi_s, \phi_t \rangle\!\rangle$ and by (5.32) we have $\|Z_-(\mu t, t)\|^2 = 0$, so $\|Z_-(s,t)\|^2 = 0$ for all $(s,t) \in \Omega_\mu^{T_*} \cap \{t \geq s\}$. Thus we have shown $\|Z_\pm(s,t)\|^2 = 0$ for all $(s,t) \in \Omega_\mu^{T_*}$ as desired.

Finally, since $\|\phi_t \pm \phi_s\|^2 = 0$ and $\phi_{tt} - \phi_{ss} = 0$ it follows that ϕ is a timelike maximal immersion conformal with respect to the metric $ds^2 - dt^2$ on $\Omega_\mu^{T_*} \setminus \mathcal{K}$ where

$$\mathcal{K} = \left\{ (s,t) \in \Omega_\mu^{T_*} \colon \|\phi_s(s,t)\|^2 = 0 \right\} \tag{5.33}$$

and the fact that

$$\mathcal{K} = \left\{ (s,t) \in \Omega_\mu^{T_*} \colon \mathrm{rank}(d\phi) \neq 2 \right\} \tag{5.34}$$

is an equivalent definition follows from the identity $\phi_s(s,t) = \frac{1}{2}(Z_+(s,t) - Z_-(s,t))$ together with the fact that the sum of two null vectors is null iff the vectors are linearly dependent. $\square$

The following is an immediate consequence of Proposition 5.5.

Corollary 5.6. *For any $C^2 \times C^1 \times C^2$ initial-boundary data (C, V, Γ) satisfying the C^2 compatibility condition* (5.15) *there is a closed subset $\Sigma \subseteq \mathbb{R}^{1+2}$ contained in the future of* $\mathrm{Im}(C)$ *with $\partial\Sigma = \mathrm{Im}(\Gamma) \cup \mathrm{Im}(C)$ and a (possibly empty) closed subset $\Sigma_{\mathrm{sing}} \subseteq \Sigma$ of Hausdorff dimension < 2 such that $\Sigma \setminus \Sigma_{\mathrm{sing}}$ is a C^2 immersed timelike maximal surface-with-boundary which contains an open neighbourhood of $\partial\Sigma$ in Σ and with V tangent to $\Sigma \setminus \Sigma_{\mathrm{sing}}$ along C.*

Proof. Let $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ be the evolution of (C, V, Γ) by isothermal gauge as in Definition 5.4. Let $\mathcal{K}$ be as in (5.26) and write

$$\Sigma = \phi(\Omega_\mu^{T_*})$$
$$\Sigma_{\text{sing}} = \phi(\mathcal{K}).$$

By Proposition 5.5 we have that ϕ is proper so Σ is closed, and ϕ is a timelike maximal immersion on $\Omega_\mu^{T_*} \setminus \mathcal{K}$ so $\Sigma \setminus \Sigma_{\text{sing}}$ is an immersed timelike maximal surface. We know that $\Omega_\mu^{T_*} \setminus \mathcal{K}$ contains a neighbourhood of $\partial\Omega_\mu^{T_*}$ because the timelikeness of Γ implies $\|\frac{d}{dt}(\phi(\mu t, t))\|^2 = (\mu^2 - 1)\|\phi_s(\mu t, t)\|^2 \neq 0$ for all $t \in [0, \infty)$ whilst the timelikeness of V implies $\|\phi_s(s, 0)\|^2 = -\|\phi_t(s, 0)\|^2 \neq 0$ for all $s \in [0, \infty)$. So $\Sigma \setminus \Sigma_{\text{sing}}$ contains a neighbourhood of $\partial\Sigma$ by continuity and by construction we have $\partial\Sigma = \operatorname{Im}(\Gamma) \cup \operatorname{Im}(C)$ with V tangent to $\Sigma \setminus \Sigma_{\text{sing}}$ along C. That Σ_{sing} has Hausdorff dimension < 2 follows from Sard's theorem. $\square$

Remark 5.7 (Higher regularity)**.** Suppose that $\mathcal{K} = \emptyset$ so that $\Sigma = \operatorname{Im}(\phi)$ is a C^2 immersed timelike maximal surface. We may decompose Σ as $\Sigma = \Sigma_+ \cup \Sigma_- \cup \mathcal{N}$ where

$$\Sigma_+ = \phi\left(\Omega_\mu^{T_*} \cap \{s > t\}\right); \, \Sigma_- = \phi\left(\Omega_\mu^{T_*} \cap \{s < t\}\right); \, \mathcal{N} = \phi\left(\Omega_\mu^{T_*} \cap \{s = t\}\right).$$

If the initial data (C, V, Γ) is $C^k \times C^{k-1} \times C^k$ for $k \geq 3$, then Σ_+ and Σ_- will be C^k by (5.7). However, Σ need not be better than C^2 across the null curve $\mathcal{N}$ which emanates from the corner $\phi(0, 0)$. In order to obtain higher regularity across $\mathcal{N}$ it is necessary to impose further compatibility conditions on the data. This is an illustration of the propagation of singularities by wave equations, and contrasts with the regularity of minimal surfaces in Euclidean space.

5.4 Analysis of the singular set

Let (C, V, Γ) be an initial-boundary data satisfying the C^2 compatibility condition (5.15) and let $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ be the evolution of (C, V, Γ) by isothermal gauge as in Definition 5.4. Recall from Proposition 5.5 that ϕ is a C^2 proper timelike maximal immersion (a global solution to IBVP 5.1) iff the singular set

$$\mathcal{K} = \Big\{(s,t) \in \Omega_\mu^{T_*} : \operatorname{rank}(d\phi) \neq 2\Big\} = \Big\{(s,t) \in \Omega_\mu^{T_*} : \|\phi_s(s,t)\|^2 = 0\Big\} \tag{5.35}$$

is empty.

Note that, in the "good" region $\{s \geq t\}$, the evolution by isothermal gauge is of the form $\phi(s,t) = (t, \gamma(s,t))$, and is indistinguishable from that considered in Chapter 4 (for the initial value problem). Thus the analysis of §4.2 may be applied directly to the set $\mathcal{K} \cap \{s \geq t\}$. In this section we will extend the analysis of §4.2 to a certain class of singular points $p \in \mathcal{K} \cap \{t > s\}$. In particular, we will prove the following result, which should be noted as suboptimal in comparison with Lemma 4.3.

Lemma 5.8. *Let* (C, V, Γ) *be a* $C^2 \times C^1 \times C^2$ *initial-boundary data satisfying the* C^2 *compatibility condition* (5.15)*, let* $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ *be the evolution of* (C, V, Γ) *by isothermal gauge as in Definition 5.4 and let* $\mathcal{K}$ *be the singular set as in* (5.35)*. For* $(s_0, t_0) \in \mathbb{R}^2$ *and* $\varepsilon > 0$ *denote*

$$L_-^\varepsilon(s_0, t_0) = \Big\{t_0 - \varepsilon < t < t_0, |s - s_0| < t_0 - t\Big\} \subseteq \mathbb{R}^2. \tag{5.36}$$

for the backward characteristic triangle of height ε *emanating from the point* (s_0, t_0)*. Suppose that there exists* $q \in \mathcal{K}$ *and* $\varepsilon > 0$ *such that*

$$\Big(\overline{L_-^\varepsilon(q)} \setminus \{q\}\Big) \cap \mathcal{K} = \emptyset.$$

Then for every neighbourhood U of q, the set $\phi(U)$ is not a subset of any C^2 immersed surface in $\mathbb{R}^{1+2}$.

To prove Lemma 5.8 we will apply Theorem 3.1 together with the following result (compare with Lemma 4.2).

Lemma 5.9. *Let (C, V, Γ) be a $C^2 \times C^1 \times C^2$ initial-boundary data satisfying the C^2 compatibility condition* (5.15)*, let $\phi \colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ be an evolution of (C, V, Γ) by isothermal gauge and let $\mathcal{K}$ denote the singular set* (5.26)*. Suppose that for some neighbourhood U of a point $q \in \partial\mathcal{K}$ there exists a C^1 embedded surface $\mathcal{T} \subseteq \mathbb{R}^{1+2}$ such that $\phi(U) \subseteq \mathcal{T}$. Then $\mathcal{T}$ is null at $\phi(q)$.*

Proof of Lemma 5.9. Let $\mathcal{T} \subseteq \mathbb{R}^{1+2}$ be a C^1 embedded surface such that $\phi(U) \subseteq \mathcal{T}$ for some neighbourhood U of a point $q \in \partial\mathcal{K}$. For all $(s, t) \in \Omega_\mu^{T_*}$, we have null vectors given by $Z_\pm(s, t) = \phi_t(s, t) \pm \phi_s(s, t)$ and these null vectors span the tangent space $T_{\phi(s,t)}\mathcal{T}$ provided $(s, t) \in U \setminus \mathcal{K}$. To prove the lemma we will first show that the vectors $Z_\pm(s, t)$ are non-zero for all $(s, t) \in \Omega_\mu^{T_*}$ and we will then argue exactly as in the proof of Lemma 4.2.

Let us show that the null vectors $Z_\pm(s, t)$ are non-zero for all $(s, t) \in \Omega_\mu^{T_*}$. Indeed, since $(\partial_t \mp \partial_s)Z_\pm = 0$, in the region $\Omega_\mu^{T_*} \cap \{s \geq t\}$ we obtain the formulas

$$Z_+(s, t) = Z_+(s + t, 0) = A_+(s + t) \tag{5.37}$$

$$Z_-(s, t) = Z_-(s - t, 0) = A_-(s - t) \tag{5.38}$$

whilst in the region $\Omega_\mu^{T_*} \cap \{t > s\}$ we obtain

$$Z_+(s,t) = Z_+(s+t,0) = A_+(s+t) \tag{5.39}$$

$$\begin{aligned} Z_-(s,t) &= Z_-(\mu\eta,\eta) = 2\phi_t(\mu\eta,\eta) - Z_+(\mu\eta,\eta) \\ &= 2\phi_t(\mu\eta,\eta) - Z_+((1+\mu)\eta,0) \\ &= 2\left(\dot{z}(\eta)\frac{d\Gamma}{dx^0}(z(\eta)) - \mu\phi_s(\mu\eta,\eta)\right) - Z_+((1+\mu)\eta,0) \\ &= \frac{2}{1-\mu}\dot{z}(\eta)\frac{d\Gamma}{dx^0}(z(\eta)) - \frac{1+\mu}{1-\mu}A_+((1+\mu)\eta) \end{aligned} \tag{5.40}$$

for some $\eta \in (0,\infty)$ where $A_\pm(s) = V(s) \pm C'(s)$ and where we appealed to d'Alembert's formula (5.7) to evaluate $\phi_s(\mu\eta,\eta)$ in the last formula. Since $A_\pm$ are non-zero and since $A_+(\cdot)$ is null whilst $\dot{z}(\cdot)\frac{d\Gamma}{dx^0}(z(\cdot))$ is non-zero and timelike we deduce from (5.37)–(5.40) that the null vectors $Z_\pm(s,t)$ are non-zero for all $(s,t) \in \Omega_\mu^{T_*}$ as claimed.

Now, take a sequence of points $q_n \in U \setminus \mathcal{K}$ with $q_n \to q$. For each q_n the tangent space $T_{\phi(q_n)}\mathcal{T}$ to $\mathcal{T}$ at $\phi(q_n)$ is a timelike plane spanned by the distinct null vectors $Z_+(q_n)$ and $Z_-(q_n)$. Since $\mathcal{T}$ is a C^1 embedded surface, the limit $\lim_{q_n\to q} T_{\phi(q_n)}\mathcal{T} = T_{\phi(q)}\mathcal{T}$ exists. Since $\mathcal{K}$ is given by (5.26) we have that $Z_+(q) = \lim_{q_n\to q} Z_+(q_n)$ and $Z_-(q) = \lim_{q_n\to q} Z_-(q_n)$ are non-vanishing and linearly dependent, so the null lines along which the tangent planes $T_{\phi(q_n)}\mathcal{T}$ intersect the light cone converge as $q_n \to q$. It follows that $T_{\phi(q)}\mathcal{T}$ is a null plane, which proves the lemma. $\square$

We will now give the proof of Lemma 5.8.

Proof of Lemma 5.8. Let $q = (s_0,t_0) \in \partial\mathcal{K}$ and $\varepsilon > 0$ be such that

$$\left(\overline{L_-^\varepsilon(q)} \setminus \{q\}\right) \cap \mathcal{K} = \emptyset$$

where $L_-^\varepsilon(q)$ is as in (5.36) and suppose for a contradiction that for some neighbourhood U of q there exists a C^2 immersed surface $\mathcal{T} \subseteq \mathbb{R}^{1+2}$ such that $\phi(U) \subseteq \mathcal{T}$. For simplicity we will assume that $\mathcal{T}$ is embedded. If $\mathcal{T}$ is not embedded then we can just run the argument below on a single leaf of $\mathcal{T}$.

By Lemma 5.9 it follows that $\mathcal{T}$ is null at $\phi(q)$. Then by the implicit function theorem there exists a neighbourhood W of $\phi(q)$ in $\mathcal{T}$ and a C^2 diffeomorphism

$$\psi\colon (-\delta_0, \delta_0) \times (\tau_0 - \delta_0, \tau_0 + \delta_0) \to W$$

of the form

$$\psi(\xi, \tau) = (\tau, \gamma(\xi, \tau))$$

for some $\delta_0 > 0$ where

$$\psi(0, \tau_0) = \phi(q). \tag{5.41}$$

By taking $\varepsilon > 0$ smaller if necessary we may assume that $\phi(L_-^\varepsilon(q))$ is contained in W.

To prove the Lemma we will show that there exists a $\xi_0\colon [\tau_0 - \delta_1, \tau_0] \to (-\delta_0, \delta_0)$ for some $\delta_1 \in (0, \delta_0)$ with $\xi_0(\tau_0) = 0$ such that the curve $\tau \mapsto \psi(\xi_0(\tau), \tau)$ is contained in $\phi(L_-^\varepsilon(q))$ for $\tau \in [\tau_0 - \delta_1, \tau_0)$ and such that the tangent vector $\frac{d}{d\tau}\psi(\xi_0(\tau), \tau)$ is orthogonal to the cross section $\{x^0\big|_W = \tau\}$ for all $\tau \in [\tau_0 - \delta_1, \tau_0)$. We will then apply Theorem 3.1 to deduce that

$$\int_{\tau_0 - \delta_1}^{\tau_0} |k(\xi_0(\tau), \tau)| d\tau = \infty$$

where $k(\cdot, \tau)$ denotes the curvature of the planar cross section $\gamma(\cdot, \tau)$, giving the desired contradiction.

To this end, let

$$i_+ \colon [t_0 - \varepsilon, t_0) \to \Omega_\mu^{T_*}; \quad i_+(t) = (s_0 + t_0 - t, t)$$

and

$$i_- \colon [t_0 - \varepsilon, t_0) \to \Omega_\mu^{T_*}; \quad i_-(t) = (s_0 - t_0 + t, t)$$

be curves which trace the right hand and left hand sides of the triangle $L_-^\varepsilon(q)$ respectively. Since ϕ is an immersion on $\overline{L_-^\varepsilon(q)} \setminus \{q\}$ and $\partial_t \phi^0 > 0$, it follows that $\phi \circ i_+$ and $\phi \circ i_-$ are C^2 future-directed null curves with

$$\lim_{t \uparrow t_0} \Big(\phi \circ i_+\Big)(t) = \lim_{t \uparrow t_0} \Big(\phi \circ i_-\Big)(t) = \phi(q). \tag{5.42}$$

Since ϕ is a C^2 timelike immersion on $\overline{L_-^\varepsilon(q)} \setminus \{q\}$ it follows that $\phi\Big(\overline{L_-^\varepsilon(q)} \setminus \{q\}\Big) \subseteq W$ is a C^2 timelike surface, so there exists a pair of C^1 future-directed null tangent vector fields N_+ and N_- on $\phi\Big(\overline{L_-^\varepsilon(q)} \setminus \{q\}\Big)$ which provide a frame for the tangent bundle of $\phi\Big(\overline{L_-^\varepsilon(q)} \setminus \{q\}\Big)$. We then have that after reparametrisation (and after relabelling N_+ and N_- if necessary) the curve $\phi \circ i_+$ is an integral curve of the vector field N_- and the curve $\phi \circ i_-$ is an integral curve of the vector field N_+. In other words, we have

$$\frac{d}{dt}\Big(\phi \circ i_+\Big)(t) = \lambda_+(t) N_-\Big(\phi \circ i_-(t)\Big) \tag{5.43}$$

$$\frac{d}{dt}\Big(\phi \circ i_-\Big)(t) = \lambda_-(t) N_+\Big(\phi \circ i_+(t)\Big) \tag{5.44}$$

for all $t \in [t_0 - \varepsilon, t_0)$ for some $\lambda_\pm \colon [t_0 - \varepsilon, t_0) \to (0, \infty)$.

Now, let the null curves $\phi \circ i_+$ and $\phi \circ i_-$ be parametrised in the (ξ, τ) coordinates as $\tau \mapsto (\xi_+(\tau), \tau)$ and $\tau \mapsto (\xi_-(\tau), \tau)$ respectively where $\xi_\pm \colon [\tau_0 - \delta_\pm, \tau_0) \to [-\delta_0, \delta_0]$ for some $\delta_\pm \in (0, \delta_0]$. By (5.41)–(5.42) we have

$$\lim_{\tau \uparrow \tau_0} \xi_+(\tau) = \lim_{\tau \uparrow \tau_0} \xi_-(\tau) = 0 \tag{5.45}$$

and from (5.43)–(5.44) it follows that the quantities $\left\langle \frac{d}{d\tau}\gamma\Big(\xi_+(\tau), \tau\Big), \gamma_\xi\Big(\xi_+(\tau), \tau\Big)\right\rangle$ and $\left\langle \frac{d}{d\tau}\gamma\Big(\xi_-(\tau), \tau\Big), \gamma_\xi\Big(\xi_-(\tau), \tau\Big)\right\rangle$ are non-vanishing and satisfy

$$\operatorname{sgn}\left\langle \frac{d}{d\tau}\gamma\Big(\xi_+(\tau), \tau\Big), \gamma_\xi\Big(\xi_+(\tau), \tau\Big)\right\rangle = -\operatorname{sgn}\left\langle \frac{d}{d\tau}\gamma\Big(\xi_-(\tau), \tau\Big), \gamma_\xi\Big(\xi_-(\tau), \tau\Big)\right\rangle$$

for all $\tau \in [\tau_0 - \min\{\delta_+, \delta_-\}, \tau_0)$. Reparametrising $\xi \mapsto -\xi$ as necessary we may take that

$$\begin{gathered}\left\langle \frac{d}{d\tau}\gamma\Big(\xi_+(\tau), \tau\Big), \gamma_\xi\Big(\xi_+(\tau), \tau\Big)\right\rangle < 0 \\ \left\langle \frac{d}{d\tau}\gamma\Big(\xi_-(\tau), \tau\Big), \gamma_\xi\Big(\xi_-(\tau), \tau\Big)\right\rangle > 0\end{gathered}$$

which implies

$$\frac{d\xi_+}{d\tau}(\tau) < \frac{-\Big\langle \gamma_\xi\Big(\xi_+(\tau), \tau\Big), \gamma_\tau\Big(\xi_+(\tau), \tau\Big)\Big\rangle}{\Big|\gamma_\xi\Big(\xi_+(\tau, \tau), \tau\Big)\Big|^2} \tag{5.46}$$

$$\frac{d\xi_-}{d\tau}(\tau) > \frac{-\Big\langle \gamma_\xi\Big(\xi_-(\tau), \tau\Big), \gamma_\tau\Big(\xi_-(\tau), \tau\Big)\Big\rangle}{\Big|\gamma_\xi\Big(\xi_-(\tau, \tau), \tau\Big)\Big|^2} \tag{5.47}$$

for all $\tau \in [\tau_0 - \min\{\delta_+, \delta_-\}, \tau_0)$.

We now observe that the curve $\tau \mapsto \psi(\xi_0(\tau), \tau)$ will run orthogonal to the cross sections $\{x^0\big|_W = \tau\}$ provided ξ_0 satisfies

$$\begin{aligned}
\frac{d\xi_0}{d\tau}(\tau) &= \frac{-\Big\langle \gamma_\xi\Big(\xi_0(\tau),\tau\Big), \gamma_\tau\Big(\xi_0(\tau),\tau\Big)\Big\rangle}{\Big|\gamma_\xi\Big(\xi_0(\tau,\tau),\tau\Big)\Big|^2} \\
\xi_0(\tau_0) &= 0,
\end{aligned} \tag{5.48}$$

so let $\xi_0 \colon [\tau_0 - \delta_1, \tau_0] \to [-\delta_0, \delta_0]$ for some $\delta_1 \in (0, \min\{\delta_+, \delta_-\}]$ denote the unique C^2 solution to the terminal value problem (5.48). From (5.42) and (5.46)–(5.48), by comparison principle we then arrive at the identity

$$\xi_-(\tau) < \xi_0(\tau) < \xi_+(\tau)$$

for all $\tau \in [\tau_0 - \delta_1, \tau_0)$. So the curve $\tau \mapsto \psi(\xi_0(\tau), \tau)$ is contained in $\phi(L_-^\varepsilon(q))$ as desired, and we will now proceed to apply Theorem 3.1 and complete the proof.

Introduce new coordinates $(\tilde{\xi}, \tilde{\tau})$ on a domain Ω defined by

$$\Omega = \Big\{\tau_0 - \delta_1 < \tilde{\tau} < \tau_0, \xi_-\Big(\tilde{\tau}\Big) - \xi_0\Big(\tilde{\tau}\Big) < \tilde{\xi} < \xi_+\Big(\tilde{\tau}\Big) - \xi_0\Big(\tilde{\tau}\Big)\Big\} \subseteq \mathbb{R}^2$$

and define a new parametrisation $\tilde{\psi} \colon \bar{\Omega} \to W$ by

$$\tilde{\psi}\Big(\tilde{\xi}, \tilde{\tau}\Big) := \Big(\tilde{\tau}, \tilde{\gamma}\Big(\tilde{\xi}, \tilde{\tau}\Big)\Big) := \Big(\tilde{\tau}, \gamma\Big(\xi_0(\tilde{\tau}) + \tilde{\xi}\Big)\Big).$$

By (5.48) we see that $\Big\langle \tilde{\gamma}_{\tilde{\xi}}\Big(0, \tilde{\tau}\Big), \tilde{\gamma}_{\tilde{\tau}}\Big(0, \tilde{\tau}\Big)\Big\rangle = 0$ and since ψ is null at the point $(0, \tau_0)$ it follows that $\Big|\tilde{\gamma}_{\tilde{\tau}}(0, \tau_0)|^2 = 1$, so $\tilde{\psi}$ satisfies the conditions for Theorem 3.1. Writing $k(\cdot, \tau)$ for the curvature of the cross section $\gamma(\cdot, \tau)$ and by Theorem 3.1 we have

$$\int_{\tau_0 - \delta_2}^{\tau_0} |k(\xi_0(\tau), \tau)| d\tau = \infty.$$

But then ψ is not C^2, giving the desired contradiction. The Lemma is proved. $\square$

5.5 Examples with $T_* = \infty$ and $T_* < \infty$

The following example is trivial, but it gives some intuition as to the geometry underlying the cases $T_* = \infty$ and $T_* < \infty$.

Example 5.10 (An example illustrating the cases $T_* = \infty$ and $T_* < \infty$)**.** Let $f\colon [0,\infty) \to \mathbb{R}$ be a smooth function which satisfies $f(0) = f'(0) = f''(0) = 0$ and $-1 < f'(x^0) \leq 0$ for all $x^0 \in [0,\infty)$ and consider an initial-boundary data (C, V, Γ) where $\Gamma\colon [0,\infty) \to \mathbb{R}^{1+2}$ is defined by

$$\Gamma(x^0) = (x^0, f(x^0), 0),$$

where $C\colon [0,\infty) \to \mathbb{R}^{1+2}$ is given by $C(s) = (0, s, 0)$ and where $V(s) = (1, 0, 0)$ for all $s \in [0,\infty)$. Since $\mathrm{Im}(\Gamma)$, $\mathrm{Im}(C)$ and $\mathrm{Im}(V)$ are contained in the timelike plane $\{x^2 = 0\} \subseteq \mathbb{R}^{1+2}$, the solution to the IBVP is obvious in this case. It is just the subset of the timelike plane given by

$$\Sigma = \{x^0 \geq 0, x^1 \geq f(x^0)\} \subseteq \{x^2 = 0\} \subseteq \mathbb{R}^{1+2}.$$

Let us compute the maximal interval of existence $[0, T_*)$ for the equation (5.20). We have $\mu = 0$ and $A_+(s) = (1, 1, 0)$ for all $s \in [0,\infty)$, so (5.20) reads

$$\begin{aligned} \frac{dz}{dt}(t) &= \frac{\langle\!\langle \frac{d\Gamma}{dx^0}(z(t)), A_+(t) \rangle\!\rangle}{\| \frac{d\Gamma}{dx^0}(z(t)) \|^2} = \frac{1}{1 + f'(z(t))}, \\ z(0) &= 0 \end{aligned}$$

which integrates to

$$z(t) + f(z(t)) = t.$$

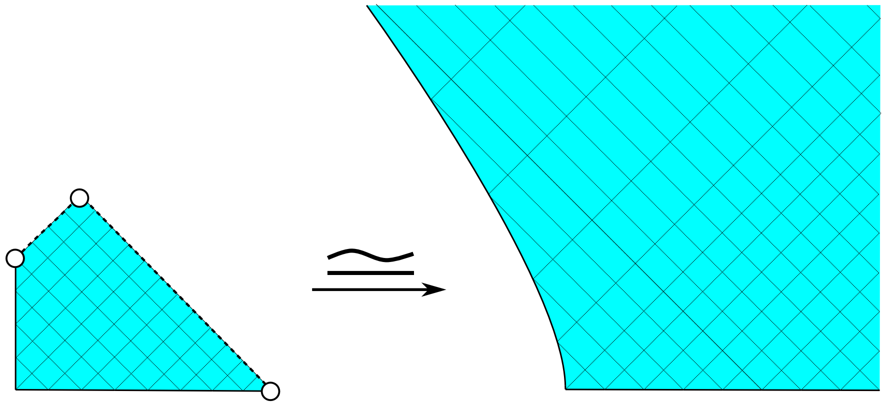


Figure 5.3: A smooth conformal equivalence between two subsets of $\mathbb{R}^{1+1}$. The right hand domain is a closed subset of $\mathbb{R}^{1+1}$ whose timelike boundary is not uniformly timelike, and the left hand domain is smoothly conformally equivalent to Ω_0^1 as in (5.4).

So we deduce that $T_* < \infty$ iff $\lim_{x^0 \uparrow \infty}(x^0 + f(x^0)) < \infty$ i.e. iff the timelike curve $x^0 \mapsto \Gamma(x^0) = (x^0, f(x^0), 0)$ asymptotes to a null line of the form $x^0 \mapsto (x^0, b - x^0, 0)$ for some $b \in (0, \infty)$ as $x^0 \uparrow \infty$.

Some geometric thinking quickly sheds light on this situation. The evolution by isothermal gauge $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ of (C, V, Γ) is a conformal equivalence which maps the null lines $\{s + t = \text{constant}\}$ and $\{s - t = \text{constant}\}$ onto null lines in $\Sigma = \operatorname{Im}(\phi)$, which are just null lines in $\mathbb{R}^{1+2}$ since $\operatorname{Im}(\phi)$ is a subset of the plane $\{x^2 = 0\}$. If the timelike boundary curve $x^0 \mapsto \Gamma(x^0) = (x^0, f(x^0), 0)$ asymptotes to a null line of the form $x^0 \mapsto (x^0, b - x^0, 0)$ for some $b \in (0, \infty)$ as $x^0 \uparrow \infty$, then it follows that the null curves in $\operatorname{Im}(\phi)$ emanating from $C(s)$ for $s \geq b$ never reach the boundary curve $\operatorname{Im}(\Gamma)$, which is consistent with the conformal structure of Ω_0^1. See Figure 5.3 for an illustration of this.

5.6 A (non-perturbative) sufficient condition on the initial-boundary data for global existence

It follows from Proposition 5.5 together with Lemma 5.8 that that the condition

$$\|\phi_s(s,t)\|^2 \neq 0 \tag{5.49}$$

for all $(s,t) \in \Omega_\mu^{T_*}$ is a necessary and sufficient condition for the evolution by isothermal gauge of a $C^2 \times C^1 \times C^2$ initial-boundary data (C, V, Γ) to give a C^2 global solution to the IBVP. Moreover, in principle at least, this condition may be checked in terms of the initial-boundary data from the d'Alembert formula (5.7). Indeed, writing $A_\pm(s) = \phi_t(s,0) \pm \phi_s(s,0)$, from (5.7) and (5.20) we may compute that

$$\phi_s(s,t) = \begin{cases} \frac{1}{2}\left(A_+(s+t) - A_-(s-t)\right) & \text{for } s \geq t \\ \frac{1}{2}\left(A_+(s+t) + A_+(s-t)\right) & \\ \qquad -\frac{1+\mu}{1-\mu}\mathbf{P}_{\left\langle \frac{d\Gamma}{dx^0}\left(z\left(\frac{t-s}{1-\mu}\right)\right)\right\rangle}\left(A_+\left(\frac{1+\mu}{1-\mu}(t-s)\right)\right) & \text{for } t > s \end{cases} \tag{5.50}$$

where

$$\mathbf{P}_{\left\langle \frac{d\Gamma}{dx^0}\left(z\left(\frac{t-s}{1-\mu}\right)\right)\right\rangle}\left(A_+\left(\frac{1+\mu}{1-\mu}(t-s)\right)\right) := \frac{\left\langle\!\left\langle \frac{d\Gamma}{dx^0}\left(z\left(\frac{t-s}{1-\mu}\right)\right), A_+\left(\frac{1+\mu}{1-\mu}(t-s)\right)\right\rangle\!\right\rangle}{\left\|\frac{d\Gamma}{dx^0}\left(z\left(\frac{t-s}{1-\mu}\right)\right)\right\|^2}\frac{d\Gamma}{dx^0}\left(z\left(\frac{t-s}{1-\mu}\right)\right)$$

is the orthogonal projection of the null vector $A_+\left(\frac{1+\mu}{1-\mu}(t-s)\right)$ onto the timelike line span$\left\{\frac{d\Gamma}{dx^0}\left(z\left(\frac{t-s}{1-\mu}\right)\right)\right\}$. In practice, however, this formula may not be given explicitly and it is hard to check the condition (5.49) in the region $t > s$ since in general the equation (5.20) for z cannot be solved by hand. The following remark gives some preliminary heuristics, nonetheless.

Remark 5.11. Since the sum of two null vectors is null iff the vectors are linearly independent, from (5.50) we see that ϕ will be non-singular in the region $s \geq t$ iff $A_+(\xi)$ and $A_-(\eta)$ are linearly independent for all $\xi \geq \eta \geq 0$. In the region $t > s$, on the other hand, note that singularity formation is governed solely by the outgoing null direction A_+. In particular, suppose the initial data (C, V) is such that $A_+(s) = A_0$ for some fixed null vector A_0 for all $s \in [0, \infty)$. Then since the sum of a timelike vector and a null vector is never null, it follows from (5.50) that $\|\phi_s(s,t)\|^2 \neq 0$ for all $t > s$ so ϕ will be non-singular in the region $t > s$ for any choice of boundary curve Γ in this case.

In the following proposition we will give a condition on the initial-boundary data (C, V, Γ) which is sufficient to ensure that the evolution by isothermal gauge is non-singular and which is more easily checked by hand than (5.49).

Proposition 5.12. *Let (C, V, Γ) be a $C^2 \times C^1 \times C^2$ initial-boundary data satisfying the C^2 compatibility condition* (5.15)*. With $C(s) = (0, c(s))$ write*

$$U_0(s) = \frac{c'(s)}{|c'(s)|} = (\cos\vartheta(s), \sin\vartheta(s))$$

for the unit-tangent vector along C, let the initial velocity be given as $V(s) = (1, v(s))$ where $\langle c'(s), v(s)\rangle = 0$, let the boundary curve $\Gamma\colon [0,\infty) \to \mathbb{R}^{1+2}$ be parameterised as $\Gamma(x^0) = (x^0, \sigma(x^0))$ and let $z\colon [0, T_*) \to [0, \infty)$ be the unique inextendible solution to (5.20)*. Suppose there exists $r \in [0, \infty)$ such that*

$$\frac{\left|\frac{d\sigma}{dx^0}(z(\eta))\right|}{1 + \left|\frac{d\sigma}{dx^0}(z(\eta))\right|} + \sup_{\xi \geq (1+\mu)\eta} \Big\{ |\sin(\vartheta(\xi) - \vartheta(r))| + |v(\xi)| \Big\} < 1 \qquad (5.51)$$

for all $\eta \in [0, T_*)$. Then the evolution $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ by isothermal gauge of (C, V, Γ)

(as in Definition 5.4) is a C^2 immersion (i.e. $\mathcal{K} = \emptyset$).

Remark 5.13 (C^1 stability of the quadrant of a timelike plane)**.** Suppose that (C, V, Γ) is an initial-boundary data satisfying the C^2 compatibility condition (5.15) such that the unit tangent $U_0(s)$ along $C(s)$ is close (in the C^0 sense) to the unit vector $(0, 1, 0)$ for all $s \in [0, \infty)$, such that $V(s) = (1, v(s))$ where $|v(s)|$ is small (in the C^0 sense) for all $s \in [0, \infty)$, and such that the unit tangent along $\Gamma(x^0)$ is close (in the C^0 sense) to the unit vector $(1, 0, 0)$ for all $x^0 \in [0, \infty)$. Then it is clear that (5.51) will be satisfied and the evolution $\phi \colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ of (C, V, Γ) by isothermal gauge will be non-singular in this case. Moreover, it may readily be checked from the d'Alembert formula that, within such a smallness regime, ϕ will be an embedding and $\mathrm{Im}(\phi)$ will be a C^2 graph over some timelike plane. This may be interpreted as a C^1 stability result for the timelike planar quadrant $\{x^0 \geq 0, x^1 \geq 0, x^2 = 0\} \subseteq \mathbb{R}^{1+2}$. An interesting point is that the conformal structure of the solution $\Omega_\mu^{T_*}$ (i.e. the conformal structure of null infinity) is preserved by this stability result.

Remark 5.14 ('End cases' of estimate (5.51))**.** Let us remark upon some borderline cases for estimate (5.51). Suppose first Γ is a straight line perpendicular to the hyper-plane $\{x^0 = 0\}$ so $\dot{\sigma} \equiv 0$. By rotating frame we may take $\vartheta(r) = 0$ so (5.51) reads

$$\sup_{\xi \in [0,\infty)} \{|\sin \vartheta(\xi)| + |v(\xi)|\} < 1.$$

As usual, writing $A_\pm(s) = (1, a_\pm(s))$ with $a_\pm(s) = \Big(\cos \alpha_\pm(s), \sin \alpha_\pm(s)\Big)$, and by trigonometry one can easily see that

$$|\sin \alpha_\pm(s)| \leq |\sin \vartheta(s)| + |v(s)| \tag{5.52}$$

so $|\sin \alpha_\pm(s)| < 1$ for all $s \in [0, \infty)$ or equivalently $\cos \alpha_\pm(s) \neq 0$ for all $s \in [0, \infty)$.

Since $\vartheta(r) = 0$, we have $\cos\alpha_+(r) > 0$ and $\cos\alpha_-(r) < 0$, and thus $\cos\alpha_+(s) > 0$ and $\cos\alpha_-(s) < 0$ for all $s \in [0,\infty)$. In other words $\mathrm{Im}(a_+)$ is contained in the right hand semi-circle and $\mathrm{Im}(a_-)$ is contained in the left hand semi-circle (we will revisit this condition in Theorem 5.15). If Γ is a straight line perpendicular to $\{x^0 = 0\}$ and $C(s) = (0, s, 0)$ (i.e. $\mathrm{Im}(C)$ is a half-line) then $\vartheta \equiv 0$, so any timelike velocity V along C is permissible (compare with Corollary 4.11) whilst, on the other hand, if $v \equiv 0$ (i.e. $V \equiv \partial_{x^0}$) we see that any curve C for which the image of the unit tangent map $\mathrm{Im}(U_0)$ is contained in an open semi-circle is permissible. In particular, writing k for the curvature of C and l for the arclength of C, by the identity $\frac{d\vartheta}{dl}(l) = k(l)$ we see that any curve C with absolute total curvature

$$\int_0^\infty |k(s)| dl(s) < \pi$$

is permissible. Next, suppose that C is a straight line and $V \equiv \partial_{x^0}$. Then (5.51) reads

$$\frac{\left|\frac{d\sigma}{dx^0}(z(\eta))\right|}{1 + \left|\frac{d\sigma}{dx^0}(z(\eta))\right|} < 1$$

for all $\eta \in [0, T_*)$ and so any timelike curve Γ is permissible. In general, condition (5.51) interpolates between the extremes described above.

Proof of Proposition 5.12. Let $z\colon [0, T_*) \to [0, \infty)$ be the inextendible solution to (5.20), $\Omega_\mu^{T_*}$ be the conformal domain (5.4), $\phi\colon \Omega_\mu^{T_*} \to \mathbb{R}^{1+2}$ be the solution of the initial-boundary problem (5.22)–(5.25), and $\mathcal{K}$ be the singular set as in (5.26). By rotating frame, we may assume without loss of generality that $\vartheta(r) = 0$ so that (5.51) reads

$$\frac{|\dot{\sigma}(z(\eta))|}{1+|\dot{\sigma}(z(\eta))|}+\sup_{\xi\geq(1+\mu)\eta}\Big\{|\sin\vartheta(\xi)|+|v(\xi)|\Big\}<1\quad\text{for all }\eta\in[0,T_*).\tag{5.53}$$

We will show that $\mathcal{K}=\emptyset$ so that ϕ is a C^2 timelike immersion and $\Sigma=\phi(\Omega_\mu^{T_*})$ gives our desired timelike maximal surface.

We first handle the region $\Omega_\mu^{T_*}\cap\{s\geq t\}$. From (5.53), we have

$$\sup_{s\in[0,\infty)}\{|\sin\vartheta(s)|+|v(s)|\}<1,\tag{5.54}$$

and we claim that (5.54) implies $\mathcal{K}\cap\{s\geq t\}=\emptyset$. Indeed, as before, writing $C(s)=(0,c(s)),V(s)=(1,v(s))$ and $a_\pm(s)=v(s)\pm c'(s)=\Big(\cos\alpha_\pm(s),\sin\alpha_\pm(s)\Big)$ for the initial null directions, from (5.52) and (5.54) we have $|\sin\alpha_\pm(s)|<1$ for all $s\in[0,\infty)$ or equivalently $\cos\alpha_\pm(s)\neq 0$ for all $s\in[0,\infty)$. Since $\vartheta(r)=0$, we have $\cos\alpha_+(r)>0$ and $\cos\alpha_-(r)<0$, and thus $\cos\alpha_+(s)>0$ and $\cos\alpha_-(s)<0$ for all $s\in[0,\infty)$ (i.e. $\mathrm{Im}(a_+)$ is contained in the right hand semi-circle and $\mathrm{Im}(a_-)$ is contained in the left hand semi-circle). So $a_+(\xi)\neq a_-(\eta)$ for all $\xi,\eta\in[0,\infty)$ which implies $\mathcal{K}\cap\{s\geq t\}=\emptyset$ as claimed.

Next, we move on to the region $\Omega_\mu^{T_*}\cap\{t\geq s\}$. Introduce the coordinates

$$\begin{aligned}\xi&=s+t\\ \eta&=\frac{t-s}{1-\mu}\end{aligned}$$

and observe $\eta \in [0, T_*)$ and $\xi \geq (1+\mu)\eta$ on $\Omega_\mu^{T_*} \cap \{t \geq s\}$. From (5.50) we have

$$\begin{aligned}
\|\phi_s(s,t)\|^2 &= \left\| \frac{1}{2}\left(A_+(\xi) + \left(\frac{1+\mu}{1-\mu}\right) A_+((1+\mu)\eta)\right) \right. \\
&\qquad \left. - \left(\frac{1+\mu}{1-\mu}\right) \frac{\langle\!\langle \frac{d\Gamma}{dx^0}(z(\eta)), A_+((1+\mu)\eta)\rangle\!\rangle}{\|\frac{d\Gamma}{dx^0}(z(\eta))\|^2} \frac{d\Gamma}{dx^0}(z(\eta)) \right\|^2 \\
&= \left(\frac{1+\mu}{1-\mu}\right) \langle\!\langle A_+(\xi), A_+((1+\mu)\eta)\rangle\!\rangle \\
&\qquad + \left(\frac{1+\mu}{1-\mu}\right)^2 \frac{\left\langle\!\!\left\langle \frac{d\Gamma}{dx^0}(z(\eta)), A_+((1+\mu)\eta)\right\rangle\!\!\right\rangle^2}{\|\frac{d\Gamma}{dx^0}(z(\eta))\|^2} \\
&\qquad\quad - \left(\frac{1+\mu}{1-\mu}\right) \left\langle\!\!\left\langle A_+(\xi) + \left(\frac{1+\mu}{1-\mu}\right) A_+((1+\mu)\eta), \frac{d\Gamma}{dx^0}(z(\eta)) \right\rangle\!\!\right\rangle \\
&\qquad\qquad \times \frac{\left\langle\!\!\left\langle \frac{d\Gamma}{dx^0}(z(\eta)), A_+((1+\mu)\eta)\right\rangle\!\!\right\rangle}{\left\|\frac{d\Gamma}{dx^0}(z(\eta))\right\|^2} \\
&= \left(\frac{1+\mu}{1-\mu}\right) \left(\langle\!\langle A_+(\xi), A_+((1+\mu)\eta)\rangle\!\rangle \right. \\
&\qquad \left. - \frac{\left\langle\!\!\left\langle A_+(\xi), \frac{d\Gamma}{dx^0}(z(\eta))\right\rangle\!\!\right\rangle \left\langle\!\!\left\langle A_+((1+\mu)\eta), \frac{d\Gamma}{dx^0}(z(\eta))\right\rangle\!\!\right\rangle}{\left\|\frac{d\Gamma}{dx^0}(z(\eta))\right\|^2} \right)
\end{aligned}$$

so the condition $\|\phi_s(s,t)\|^2 > 0$ for all $t > s$ becomes

$$\begin{aligned}
&\frac{1}{2}\langle\!\langle A_+(\xi), A_+((1+\mu)\eta)\rangle\!\rangle \|\frac{d\Gamma}{dx^0}(z(\eta))\|^2 \\
&\qquad - \langle\!\langle A_+(\xi), \frac{d\Gamma}{dx^0}(z(\eta))\rangle\!\rangle \langle\!\langle A_+((1+\mu)\eta), \frac{d\Gamma}{dx^0}(z(\eta))\rangle\!\rangle < 0
\end{aligned}$$

for all $\eta \in [0, T_*), \xi \geq (1+\mu)\eta$ which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\Big(1 - \langle a_+(\xi), a_+((1+\mu)\eta)\rangle\Big) \\
&\qquad < \frac{(1 - \langle a_+(\xi), \frac{d\sigma}{dx^0}(z(\eta))\rangle)(1 - \langle a_+((1+\mu)\eta), \frac{d\sigma}{dx^0}(z(\eta))\rangle)}{1 - \left|\frac{d\sigma}{dx^0}(z(\eta))\right|^2}
\end{aligned} \tag{5.55}$$

for all $\eta \in [0, T_*), \xi \geq (1+\mu)\eta$. In particular, by Cauchy-Schwarz and since $|a_+| = 1$, observe that (5.55) will be satisfied provided

$$\frac{1}{2}\left(1-\langle a_+(\xi), a_+((1+\mu)\eta)\rangle\right) < \frac{(1-\left|\frac{d\sigma}{dx^0}(z(\eta))\right|)^2}{1-\left|\frac{d\sigma}{dx^0}(z(\eta))\right|^2} = \frac{1-\left|\frac{d\sigma}{dx^0}(z(\eta))\right|}{1+\left|\frac{d\sigma}{dx^0}(z(\eta))\right|}$$

for all $\eta \in [0, T_*), \xi \geq (1+\mu)\eta$, and we deduce that $\|\phi_s(s,t)\|^2 > 0$ for all $t > s$ provided

$$\frac{\left|\frac{d\sigma}{dx^0}(z(\eta))\right|}{1+\left|\frac{d\sigma}{dx^0}(z(\eta))\right|} + \sup_{\xi \geq (1+\mu)\eta} \frac{1}{2}\left(1-\langle a_+(\xi), a_+((1+\mu)\eta)\rangle\right) < 1 \tag{5.56}$$

for all $\eta \in [0, T_*)$. Thus to see that $\mathcal{K} \cap \{t \geq s\} = \emptyset$ it suffices to show that (5.56) is satisfied.

Denote by

$$\omega(\xi, \eta) = \arccos\langle a_+(\xi), a_+((1+\mu)\eta)\rangle$$

the angle between the vectors $a_+(\xi)$ and $a_+((1+\mu)\eta)$. Since $\mathrm{Im}(a_+)$ is contained in the right hand semi-circle, it follows that

$$\frac{|\omega(\xi,\eta)|}{2} \leq \sup_{\xi \geq (1+\mu)\eta} |\alpha_+(\xi)| < \frac{\pi}{2}$$

for $\xi \geq (1+\mu)\eta$. From (5.52) we then arrive at

$$\begin{aligned}
\frac{1}{2}(1-\langle a_+(\xi), a_+((1+\mu)\eta)\rangle) &= \frac{1}{2}(1-\cos\omega(\xi,\eta)) = \sin^2\frac{\omega(\xi,\eta)}{2} \\
&\leq \sup_{\xi \geq (1+\mu)\eta} \sin^2(\alpha_+(\xi)) \\
&\leq \sup_{\xi \geq (1+\mu)\eta} |\sin(\alpha_+(\xi))| \\
&\leq \sup_{\xi \geq (1+\mu)\eta} \{|\sin\vartheta(\xi)| + |v(\xi)|\},
\end{aligned} \tag{5.57}$$

and (5.56) follows from (5.51) and (5.57). Thus $\mathcal{K} \cap \{t > s\} = \emptyset$ and the Proposition is proved. $\square$

5.7 The case where the timelike boundary curve is a straight line

In this section we will analyse the special case that the timelike boundary curve Γ is a half-line. Choosing inertial coordinates (x^0, x^1, x^2) on $\mathbb{R}^{1+2}$ appropriately. without loss of generality we may take Γ to be given by $\Gamma\colon [0,\infty) \to \mathbb{R}^{1+2}$, $\Gamma(x^0) = (x^0, 0, 0)$. This case is particularly simple to analyse as the differential equation (5.20) may be solved explicitly.

Theorem 5.15. *Let (C, V, Γ) be a $C^2 \times C^1 \times C^2$ initial-boundary data satisfying the C^2 compatibility condition* (5.15) *where the boundary curve $\Gamma\colon [0,\infty) \to \mathbb{R}^{1+2}$ is the half-line $\Gamma(x^0) = (x^0, 0, 0)$, let $\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ be the evolution of (C, V, Γ) by isothermal gauge and write $A_\pm(s) = (1, a_\pm(s))$ for the future-directed null vectors spanning the tangent space* $\operatorname{span}\Big\{C'(s), V(s)\Big\}$. *Then ϕ is an immersion iff $\operatorname{Im}(a_+)$ is contained in an open semi-circle and $a_+(\xi) \neq a_-(\eta)$ for all $\xi \geq \eta \geq 0$.*

Proof. The evolution of (C, V, Γ) by isothermal gauge may be computed as

$$\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}; \quad \phi(s,t) = (t, \gamma(s,t))$$

where

$$\gamma(s,t) = \begin{cases} \frac{1}{2}\Big(c(s+t) + c(s-t) + \int_{s-t}^{s+t} v(\zeta)d\zeta\Big) & \text{for } s \geq t \\ \frac{1}{2}\Big(c(s+t) - c(t-s) + \int_{t-s}^{s+t} v(\zeta)d\zeta\Big) & \text{for } t > s. \end{cases} \tag{5.58}$$

Then ϕ is an immersion iff $\gamma_s \neq 0$ and from (5.58) we have

$$\gamma_s(s,t) = \begin{cases} \frac{1}{2}\left(a_+(s+t) - a_-(s-t)\right) & \text{for } s \geq t \\ \frac{1}{2}\left(a_+(s+t) + a_+(t-s)\right) & \text{for } t > s \end{cases} \tag{5.59}$$

so ϕ is an immersion in the region $s \geq t$ iff $a_+(\xi) \neq a_-(\eta)$ for all $\xi \geq \eta \geq 0$ whilst ϕ is an immersion in the region $t > s$ iff $a_+(\xi) \neq -a_+(\eta)$ for all $\xi, \eta \in [0, \infty)$. This proves the theorem. $\square$

Remark 5.16 (Self-intersecting global solutions)**.** Just as in the case for the IVP (recall Remark 4.8 and refer to Figure 4.2) it is easy to construct an initial-bounday data (C, V, Γ) where Γ is the half-line $\Gamma(x^0) = (x^0, 0, 0)$ for which the curve C is self-intersecting and for which $\mathrm{Im}(a_+)$ is contained in an open semi circle and $a_+(\xi) \neq a_-(\eta)$ for all $\xi \geq \eta \geq 0$, so that the evolution by isothermal gauge of (C, V, Γ) is non-singular in this case (this hinges crucially on the fact that we consider here only future evolutions).

Example 5.17 (An example of singularity formation due to the "reflection of waves off the boundary")**.** Here we will give a simple example of an initial-boundary data (C, V, Γ) where the boundary curve $\Gamma\colon [0, \infty) \to \mathbb{R}^{1+2}$ is the half-line $\Gamma(x^0) = (x^0, 0, 0)$ such that the evolution ϕ by isothermal gauge of (C, V, Γ) is an immersion in the region $s \geq t$ but not an immersion in the region $t > s$. That is to say, singularity only occurs outside of the domain of dependence of the initial curve (i.e. due to contributions from the boundary).

Let $\Gamma\colon [0, \infty) \to \mathbb{R}^{1+2}$ be the half-line $\Gamma(x^0) = (x^0, 0, 0)$ and let $C\colon [0, \infty) \to \mathbb{R}^{1+2}$ be a smooth immersion $C(s) = (0, c(s))$ such that $\mathrm{Im}(\vartheta) = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ where

$$\frac{c'(s)}{|c'(s)|} = \left(\cos\vartheta(s), \sin\vartheta(s)\right) \tag{5.60}$$

denotes the unit tangent along C. We may construct a vector field V along C such that $\mathrm{Im}(a_-)$ is a single point whilst $\mathrm{Im}(a_+)$ is a closed semi-circle (where we write $A_\pm(s) = (1, a_\pm(s))$ for the null vectors spanning the plane $\mathrm{span}\{C'(s), V(s)\}$ as usual). Indeed, such a vector field is given explicitly by $V(s) = (1, \sin\vartheta(s) U_0(s)^\perp)$ where $\perp$ denotes anticlockwise rotation in the plane by $\frac{\pi}{2}$ radians. From identity (5.59) we see the evolution ϕ by isothermal gauge of (C, V, Γ) is an immersion in the region $s \geq t$ but not an immersion in the region $t > s$.

Chapter 6

Applications to further initial-boundary value problems

6.1 An IBVP (of Neumann type) for a timelike maximal surface which intersects a timelike plane orthogonally along its timelike boundary

In this section we will briefly consider an IBVP for a timelike maximal surface which meets a given timelike surface orthogonally along its timelike boundary, which may be thought of as a kind of Neumann boundary condition. Interestingly, we will see that the IBVP of this section can be reduced to the one considered in Chapter 5, via isothermal gauge considerations.

To keep the presentation simple, we will treat in this thesis only the case that the timelike boundary surface is a plane, but let us state the Neumann IBVP in more generality first. Let $\mathcal{J} \subseteq \mathbb{R}^{1+2}$ be a C^2 connected properly immersed timelike surface.

Note that, by Morse theory, $\mathcal{J}$ is either an immersed $S^1 \times \mathbb{R}$ or an immersed $\mathbb{R}^2$. Let $C\colon [0,\infty) \to \mathbb{R}^{1+2}$ be a C^2 proper immersion of the form $C(s) = (0, c(s))$ such that $C(0) \in \mathcal{J}$ and such that C meets $\mathcal{J}$ orthogonally at 0 and let V be a C^1 timelike vector field along C such that $V(0) \in T_{C(0)}\mathcal{J}$. We call the trio $(C, V, \mathcal{J})$ an initial-boundary data of Neumann type. Given such an initial-boundary data $(C, V, \mathcal{J})$, the future (resp. past) IBVP is to find a C^2 proper timelike maximal immersion

$$\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$$

such that $s \mapsto \phi(s, 0)$ is a monotone reparameterisation of C, $t \mapsto \phi(0, t)$ is a future-directed (resp. past-directed) timelike embedding into the surface $\mathcal{J}$ along which ϕ intersects $\mathcal{J}$ orthogonally, $\mathrm{Im}(\phi)$ is contained in the future (resp. past) of $\mathrm{Im}(C)$ and V is tangent to $\mathrm{Im}(\phi)$ along C.

Remark 6.1. We note that, in contrast with the IBVP of §5.1, there do not seem to be any natural C^2 compatibility conditions for an initial-boundary data of Neumann type. We will see however, that as before, a priori restrictions on the conformal structure does impose C^2 constraints on the data.

To keep the presentation simple, in this thesis we will consider only the case that the timelike boundary surface is a plane. It should be clear from what follows, however, that the same methods may be applied to treat an arbitrary timelike boundary surface. We will treat only the future IBVP since the past IBVP may be treated analogously (the map $x^0 \mapsto -x^0$ is an isometry of $\mathbb{R}^{1+2}$).

Let $\mathcal{J} \subseteq \mathbb{R}^{1+2}$ be a timelike plane. We may choose inertial coordinates (x^0, x^1, x^2) on $\mathbb{R}^{1+2}$ so that $\mathcal{J} = \{x^2 = 0\}$. Let (z_+, z_-) be null coordinates on $\mathcal{J}$ so that a parameterisation of $\mathcal{J}$ is given by $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$;

$$\Psi(z_+, z_-) = \left(\frac{z_+ + z_-}{2}, \frac{z_+ - z_-}{2}, 0\right). \tag{6.1}$$

We seek timelike maximal surfaces orthogonal to $\mathcal{J}$, and to this end we will look for solutions with the conformal structure $\Omega_0^{T_*}$ for some $T_* \in (0, \infty]$ where $\Omega_0^{T_*}$ is as defined in (5.4).

Suppose that $\phi\colon \Omega_0^{T_*} \to \mathbb{R}^{1+2}$ is a C^2 timelike maximal immersion which is conformal with respect to the metric $ds^2 - dt^2$ on $\Omega_0^{T_*}$, i.e. ϕ satisfies

$$\|\phi_t \pm \phi_s\|^2 = 0 \tag{6.2}$$

$$\phi_{tt} - \phi_{ss} = 0, \tag{6.3}$$

and which is a solution to the future IBVP of Neumann type, so that ϕ satisfies the initial-boundary conditions

$$\phi(s, 0) = C(s) = (0, c(s)) \tag{6.4}$$

$$\phi_t(s, 0) = V(s) = (1, v(s)) \tag{6.5}$$

$$\phi(0, t) = \Psi(z_+(t), z_-(t)) \tag{6.6}$$

where we assume that $C\colon [0, \infty) \to \mathbb{R}^{1+2}$ meets $\mathcal{J} = \{x^2 = 0\}$ orthogonally at 0 with $C(0) = (0, 0, 0)$, $\mathrm{Im}(\phi)$ is contained in the future of $\mathrm{Im}(C)$, and the boundary curve $t \mapsto \phi(0, t) = \Psi(z_+(t), z_-(t))$ is future-directed timelike with ϕ intersecting $\mathcal{J}$ orthogonally along $t \mapsto \phi(0, t)$.

We will now show that the boundary curve is uniquely determined by the initial data.[1] Since $\Sigma = \mathrm{Im}(\phi)$ meets $\mathcal{J}$ orthogonally along the curve $t \mapsto \phi(0, t)$, it follows

[1] It may be the case that this is a consequence of our imposition of conformal structure. At the time of writing, the author is not sure whether or not this is the case.

that the normal to $T_{\phi(0,t)}\mathcal{J} \cong \mathcal{J}$ is contained in $T_{\phi(0,t)}\Sigma$. But by (6.2) we have $\langle\!\langle \phi_s(0,t), \phi_t(0,t) \rangle\!\rangle = 0$ and so it then follows that $\phi_s(0,t)$ must be normal to $T_{\phi(0,t)}\mathcal{J}$. Taking

$$\phi_t(0,t) = \dot{z}_+(t)\frac{\partial \Psi}{\partial z_+}(z_+(t), z_-(t)) + \dot{z}_-(t)\frac{\partial \Psi}{\partial z_-}(z_+(t), z_-(t))$$
$$\phi_t(0,t)^\perp = \dot{z}_+(t)\frac{\partial \Psi}{\partial z_+}(z_+(t), z_-(t)) - \dot{z}_-(t)\frac{\partial \Psi}{\partial z_-}(z_+(t), z_-(t))$$

as an orthogonal basis for $T_{\phi(0,t)}\mathcal{J}$ we then deduce by (6.3) that

$$\begin{aligned} 0 &= \langle\!\langle \phi_s(0,t), \phi_t(0,t) \rangle\!\rangle = \langle\!\langle \phi_s(0,t) + \phi_t(0,t) - \phi_t(0,t), \phi_t(0,t) \rangle\!\rangle \\ &= \langle\!\langle \phi_s(t,0) + \phi_t(t,0) - \phi_t(0,t), \phi_t(0,t) \rangle\!\rangle = \langle\!\langle A_+(t) - \phi_t(0,t), \phi_t(0,t) \rangle\!\rangle \\ 0 &= \langle\!\langle \phi_s(0,t), \phi_t(0,t)^\perp \rangle\!\rangle = \langle\!\langle \phi_s(0,t) + \phi_t(0,t), \phi_t(0,t)^\perp \rangle\!\rangle \\ &= \langle\!\langle \phi_s(t,0) + \phi_t(t,0), \phi_t(0,t) \rangle\!\rangle = \langle\!\langle A_+(t), \phi_t(0,t)^\perp \rangle\!\rangle. \end{aligned}$$

and, after a bit of algebra, the above equations reduce to give the following equations for $(z_+, z_-)\colon [0, T_*) \to \mathbb{R}^2$

$$\dot{z}_+(t) = \frac{\left\langle\!\!\left\langle A_+(t), \frac{\partial \Psi}{\partial z_-}(z_+(t), z_-(t)) \right\rangle\!\!\right\rangle}{\left\langle\!\!\left\langle \frac{\partial \Psi}{\partial z_+}(z_+(t), z_-(t)), \frac{\partial \Psi}{\partial z_-}(z_+(t), z_-(t)) \right\rangle\!\!\right\rangle} = 1 + a_+^1(t), \tag{6.7}$$

$$\dot{z}_-(t) = \frac{\left\langle\!\!\left\langle A_+(t), \frac{\partial \Psi}{\partial z_+}(z_+(t), z_-(t)) \right\rangle\!\!\right\rangle}{\left\langle\!\!\left\langle \frac{\partial \Psi}{\partial z_+}(z_+(t), z_-(t)), \frac{\partial \Psi}{\partial z_-}(z_+(t), z_-(t)) \right\rangle\!\!\right\rangle} = 1 - a_+^1(t), \tag{6.8}$$

$$z_+(0) = z_-(0) = 0 \tag{6.9}$$

where, as usual, $A_+(s) = V(s) + C'(s) = (1, a_+(s))$. The equations (6.7)–(6.9) determine the C^2 timelike boundary curve $t \mapsto \phi(0,t) = \Psi(z_+(t), z_-(t))$ uniquely in terms of the outgoing null vector A_+ (i.e. in terms of the initial data (C, V)).

Remark 6.2 (Conformal structure of null infinity)**.** Note that, since ϕ is proper, we have $z_\pm(t) \uparrow \infty$ as $t \uparrow T_*$ and from (6.7)–(6.8) we have $|\dot{z}_\pm(t)| \leq 2$ so it follows necessarily that $T_* = \infty$. So we have shown that the conformal structure is necessarily $\Omega_0^\infty = [0,\infty) \times [0,\infty) \subseteq \mathbb{R}^{1+1}$.

Remark 6.3. The fact that the equations (6.7)–(6.8) are linear and decoupled follows since $\mathcal{J} = \{x^0 = 0\}$ is a plane. For a general curved boundary surface, one obtains a nonlinear coupled system for the boundary curve.

Next, as in §5.2, let us proceed to derive C^2 compatibility conditions on the pair (C, V) at the point 0 (i.e. at the corner). We have by (6.3) that $\lim_{t\downarrow 0} \phi_{tt}(0,t) = \lim_{s\downarrow} \phi_{ss}(s,0)$ which implies

$$\ddot{z}_+(0)\frac{\partial \Psi}{\partial z_+} + \ddot{z}_-(0)\frac{\partial \Psi}{\partial z_-} = C''(0).$$

Applying (6.1) and (6.7)–(6.8) the x^0-component of the above equation reads $\ddot{z}_+(0) + \ddot{z}_-(0) = 0$, which is seen to hold automatically from (6.7)–(6.8), whilst the x^1 and x^2 components read $(v')^1(0) = 0$ and $(c'')^2(0) = 0$ which, when expressed in terms of arclength parameter l along c, read as

$$\frac{dv^1}{dl}(0) = 0 \tag{6.10}$$

$$(1 - |v(0)|^2)\frac{d^2c^2}{dl^2}(0) - \langle v(0), \frac{dv}{dl}(0)\rangle \frac{dc^2}{dl^2}(0) = 0. \tag{6.11}$$

Let's now state formally the IBVP we are to consider here.

IBVP 6.4 (Future IBVP of Neumann type)**.** Given a $C^2 \times C^1 \times C^2$ initial-boundary data $(C, V, \mathcal{J})$ of Neumann type where $\mathcal{J}$ is the timelike plane $\mathcal{J} = \{x^2 = 0\} \subseteq \mathbb{R}^{1+2}$ and where (C, V) satisfies the C^2 compatibility conditions (6.10)–(6.11), find a C^2 proper timelike maximal immersion $\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ which is conformal with respect to the metric $ds^2 - dt^2$ on $[0,\infty) \times [0,\infty)$ such that $s \mapsto \phi(s, 0)$ is a monotone reparameterisation of C, $t \mapsto \phi(0,t)$ is a future-directed timelike curve along which ϕ intersects $\mathcal{J}$ orthogonally, and V is tangent to $\mathrm{Im}(\phi)$ along C.

Definition 6.5. Suppose that $(C, V, \mathcal{J})$ is a $C^2 \times C^1 \times C^2$ initial-boundary data of Neumann type where $\mathcal{J} = \{x^2 = 0\} \subseteq \mathbb{R}^{1+2}$ is a timelike plane which satisfies the C^2 compatibility conditions (6.10)–(6.11). Let the initial data be parameterised isothermally so that $A_\pm(s) = V(s) \pm C'(s)$ are null vectors, let $(z_+, z_-)\colon [0,\infty) \to \mathbb{R}^2$ be the unique C^2 solution to (6.7)–(6.9) and let $\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ be the unique solution to (6.3)–(6.6). Then we call ϕ the future evolution of $(C, V, \mathcal{J})$ by isothermal gauge.

It may be checked (refer to the proof of Proposition 5.5 for the main ideas) that the future evolution by isothermal gauge $\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ of Definition 6.5 is a (global) solution to IBVP 6.4 iff ϕ is an immersion. Moreover, it is readily checked that if ϕ is an immersion then ϕ intersects $\mathcal{J}$ orthogonally along the future-directed timelike curve $t \mapsto \phi(0,t)$.

Theorem 6.6. *Let $(C, V, \mathcal{J})$ is a $C^2 \times C^1 \times C^2$ initial-boundary data of Neumann type where $\mathcal{J} = \{x^2 = 0\} \subseteq \mathbb{R}^{1+2}$ is a timelike plane which satisfies the C^2 compatibility conditions* (6.10)–(6.11)*, let $\phi\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^{1+2}$ be the future evolution of $(C, V, \mathcal{J})$ by isothermal gauge as in Definition 6.5 and, as usual, write $A_\pm(s) = (1, a_\pm(s))$ for the future-directed null vectors spanning the timelike plane* $\mathrm{span}\{C'(s), V(s)\}$*. Then ϕ is a C^2 immersion iff* $\mathrm{Im}(a_+)$ *is contained in either the upper open semi-circle $\{(a^1, a^2) \in S^1 \subseteq \mathbb{R}^2\colon a^2 > 0\}$ or the lower open semi-circle $\{(a^1, a^2) \in S^1 \subseteq \mathbb{R}^2\colon a^2 < 0\}$ and $a_+(\xi) \neq a_-(\eta)$ for all $\xi \geq \eta \geq 0$.*

Remark 6.7 (Small-data global existence)**.** It is clear that if the unit tangent $U_0(s)$ along $C(s)$ is close (in the C^0 sense) to the fixed unit vector $\omega = (0,1)$ for all $s \in [0,\infty)$ and if $V(s) = (1, v(s))$ where $|v(s)|$ is small for all $s \in [0,\infty)$ (in the C^0 sense) then

$\mathrm{Im}(a_+)$ is contained in the right-hand open semi-circle and $\mathrm{Im}(a_+)$ and $\mathrm{Im}(a_-)$ are disjoint subsets of S^1 and so by Theorem 6.6 the future evolution of $(C, V, \mathcal{J})$ by isothermal gauge will be non-singular in this case. Moreover, it may be checked from a d'Alembert formula that the solution will be a C^2 graph within such a smallness regime.

Proof of Theorem 6.6. It may be seen that ϕ is of the form $\phi(s,t) = (t, \gamma(s,t))$ and from the d'Alembert formula one may compute directly that

$$\gamma_s(s,t) = \begin{cases} \frac{1}{2}\Big(a_+(s+t) - a_-(s-t)\Big) & \text{for } s \geq t \\ \frac{1}{2}\Big(a_+(s+t) + a_+(t-s)\Big) - \mathbf{p}(a_+(t-s)) & \text{for } t > s \end{cases} \tag{6.12}$$

where $\mathbf{p}(a^1, a^2) := (a^1, 0)$ denotes the orthogonal projection onto the line $\{x^2 = 0\}$ in the x^1–x^2 plane. In particular, it may be seen to follow from (6.12) that ϕ is an immersion in the region $s \geq t$ iff $a_+(\xi) \neq a_-(\eta) \geq 0$ for all $\xi \geq \eta \geq 0$ whilst ϕ is an immersion in the region $t > s$ iff $\mathrm{Im}(a_+)$ is contained in either the top open semi-circle $\{(a^1, a^2) \in S^1 \subseteq \mathbb{R}^2 \colon a^2 > 0\}$ or the bottom open semi-circle $\{(a^1, a^2) \in S^1 \subseteq \mathbb{R}^2 \colon a^2 < 0\}$. This proves the theorem. $\square$

6.2 An IBVP for a timelike maximal surface bounded between a pair of parallel timelike lines

In this section we will briefly consider an IBVP for a timelike maximal surface bounded between a given pair of timelike curves. In this thesis we will only treat the case that the boundary curves are a pair of parallel timelike lines, but let us state the problem in full generality first. Let $\Gamma_1, \Gamma_2 \colon \mathbb{R} \to \mathbb{R}^{1+2}$ be a pair of C^2 future-directed proper timelike immersions with $\Gamma_1(0), \Gamma_2(0) \in \{x^0 = 0\}$, let $C\colon [0,\lambda] \to \mathbb{R}^{1+2}$ be a C^2 immersion of the form $C(s) = (0, c(s))$ with

$$C(0) = \Gamma_1(0), \quad C(\lambda) = \Gamma_2(0), \tag{6.13}$$

and let V be a C^1 future-directed timelike vector field along C with

$$V(0) \in \operatorname{span}\left\{C'(0), \frac{d\Gamma_1}{dx^0}(0)\right\}, \quad V(\lambda) \in \operatorname{span}\left\{C'(\lambda), \frac{d\Gamma_2}{dx^0}(0)\right\}. \tag{6.14}$$

We call such $(C, V, \Gamma_1, \Gamma_2)$ an initial-boundary data. Given an initial-boundary data $(C, V, \Gamma_1, \Gamma_2)$ the IBVP is to find a C^2 proper timelike maximal immersion

$$\phi\colon [0,1] \times \mathbb{R} \to \mathbb{R}^{1+2}$$

such that $s \mapsto \phi(s, 0)$ is a monotone reparameterisation of C, $t \mapsto \phi(0, t)$ is a monotone reparameterisation of Γ_1, $t \mapsto \phi(1, t)$ is a monotone reparameterisation of Γ_2, and V is tangent to $\mathrm{Im}(\phi)$ along C. As before, a global solution is a proper immersion.

In this thesis we will treat the above IBVP only in the special case where the prescribed timelike boundary curves are a pair of parallel sraight lines. The advantage of this case is that the method of images (or, equivalently, d'Alembert's method) may be applied to give an explicit representation formula for the solution by isothermal gauge (other cases of timelike boundary data are much harder to analyse). In addition, we will look for solutions to the IBVP with the conformal structure

$$\bar{\mathbb{D}}^{1+1} = [0,1] \times \mathbb{R} \subseteq \mathbb{R}^{1+1}. \tag{6.15}$$

Remark 6.8. Note that the conformal structure $\mathbb{D}^{1+1} = (0,1) \times \mathbb{R} \subseteq \mathbb{R}^{1+1}$ bears some significance in Lorentzian geometry, and has been described by Kulkarni in the context of Lorentz surfaces as "a Lorentz analogue of [the unit disc] $\mathbb{D}^2$" [46, Introduction]. From Kulkarni's work, it may be seen to follow that any properly immersed solution to the IBVP for a pair of timelike boundary curves will have an induced metric which is C^0 conformally equivalent to $\bar{\mathbb{D}}^{1+1}$ (in Kulkarni's language, the precise statement is that a simply connected Lorentzian surface is C^0 conformally equivalent to $\mathbb{D}^{1+1}$ iff its ideal boundary is smoothable and contains no characteristic point [46, Section 3]).

Let $\Gamma_1, \Gamma_2 \colon \mathbb{R} \to \mathbb{R}^{1+2}$ be a pair of timelike parallel lines. By choosing inertial coordinates (x^0, x^1, x^2) on $\mathbb{R}^{1+2}$ appropriately and rescaling as necessary (recall that the mean curvature is scale invariant) we may take that $\Gamma_1(x^0) = (x^0, 0, 0)$, $\Gamma_2(x^0) = (x^0, 1, 0)$. We assume initial data of the form $C \colon [0, \lambda] \to \mathbb{R}^{1+2}$ where $C(s) = (0, c(s))$ and $V(s) = (1, v(s))$ and we look for a $\lambda > 0$ and a C^2 timelike maximal immersion $\phi \colon [0, \lambda] \times \mathbb{R} \to \mathbb{R}^{1+2}$ which is conformal with respect to the metric $ds^2 - dt^2$ on $[0, \lambda] \times \mathbb{R}$ and satisfies

$$\begin{aligned} \phi_{tt} - \phi_{ss} &= 0 \\ \phi(s, 0) &= (0, c(s)) \\ \phi_t(s, 0) &= (1, v(s)) \\ \phi(0, t) &= (t, 0, 0) \\ \phi(\lambda, t) &= (t, 1, 0). \end{aligned} \tag{6.16}$$

Note that the domain $[0, \lambda] \times \mathbb{R} \subseteq \mathbb{R}^{1+1}$ is obviously smoothly conformally equivalent to $\bar{\mathbb{D}}^{1+1} = [0, 1] \times \mathbb{R} \subseteq \mathbb{R}^{1+1}$ by a rescaling of $\mathbb{R}^{1+1}$. This implies the isothermal conditions $|v(s) \pm c'(s)|^2 = 1$ as well as (refer to the C^2 compatibility condition (5.15) derived in §5.2) the C^2 compatibility conditions

$$\frac{d^2c}{dl^2}(0) = \frac{d^2c}{dl^2}(\lambda) = (0,0) \tag{6.17}$$

where l denotes arclength parameter along c. We may now state formally the IBVP that we will consider here.

IBVP 6.9. Given an initial-boundary data $(C, V, \Gamma_1, \Gamma_2)$ with boundary data given by the pair of parallel lines $\Gamma_1(x^0) = (x^0, 0, 0)$, $\Gamma_2(x^0) = (x^0, 1, 0)$ and where C satisfies the C^2 compatibility condition (6.17), find a C^2 proper timelike maximal immersion $\phi\colon \bar{\mathbb{D}}^{1+1} \to \mathbb{R}^{1+2}$ where $\bar{\mathbb{D}}^{1+1}$ is as in (6.15) which is conformal with respect to the metric $ds^2 - dt^2$ on $\bar{\mathbb{D}}^{1+1}$ such that $s \mapsto \phi(s, 0)$ is a monotone reparameterisation of C, $t \mapsto \phi(0, t)$ is a monotone reparameterisation of Γ_1, $t \mapsto \phi(1, t)$ is a monotone reparameterisation of Γ_2 and V is tangent to $\mathrm{Im}(\phi)$ along C.

Definition 6.10. Let $(C, V, \Gamma_1, \Gamma_2)$ be an initial-boundary data with boundary data given by the pair of parallel lines $\Gamma_1(x^0) = (x^0, 0, 0)$, $\Gamma_2(x^0) = (x^0, 1, 0)$ and where C satisfies the C^2 compatibility condition (6.17). Let the initial data be parameterised isothermally so that $C\colon [0, \lambda] \to \mathbb{R}^{1+2}$, $C(s) = (0, c(s))$, $V(s) = (1, v(s))$ where $\langle c'(s), v(s)\rangle = 0$, $|c'(s)|^2 + |v(s)|^2 = 1$ and let $\phi\colon [0, \lambda] \times \mathbb{R} \to \mathbb{R}^{1+2}$ denote the unique C^2 solution to (6.16). Then we call ϕ the evolution of $(C, V, \Gamma_1, \Gamma_2)$ by isothermal gauge.

It may be seen (refer to the proof of Proposition 5.5 for the ideas) that the evolution by isothermal gauge $\phi\colon [0, \lambda] \times \mathbb{R} \to \mathbb{R}^{1+2}$ of an initial-boundary data $(C, V, \Gamma_1, \Gamma_2)$ as in Definition 6.10 is a (global) solution to IBVP 6.9 iff ϕ is an immersion.

Theorem 6.11. *Let $(C, V, \Gamma_1, \Gamma_2)$ be an initial-boundary data where the timelike boundary curves $\Gamma_1, \Gamma_2\colon \mathbb{R} \to \mathbb{R}^{1+2}$ are a pair of parallel straight lines $\Gamma_1(x^0) = (x^0, 0, 0)$, $\Gamma_2(x^0) = (x^0, 1, 0)$ and where the initial data satsifies the C^2 compatibility* *conditions* (6.17)*, and let* $\phi\colon [0,\lambda]\times\mathbb{R}\to\mathbb{R}^{1+2}$ *be the evolution of* $(C, V, \Gamma_1, \Gamma_2)$ *by isothermal gauge (Definition 6.10). Write* $A_\pm(s) = (1, a_\pm(s))$ *for the future-directed null vector fields along* C *such that*

$$\operatorname{span}\Big\{A_+(s), A_-(s)\Big\} = \operatorname{span}\Big\{C'(s), V(s)\Big\}.$$

Then ϕ *is a* C^2 *immersion iff there exist a pair of disjoint open semi-circles* $\Lambda_+ \subseteq S^1$ *and* $\Lambda_- \subseteq S^1$ *such that* $\operatorname{Im}(a_+) \subseteq \Lambda_+$ *and* $\operatorname{Im}(a_-) \subseteq \Lambda_-$*. Moreover if* ϕ *is a* C^2 *immersion then* ϕ *is an embedding,* $\operatorname{Im}(\phi)$ *is a* C^2 *graph over some timelike plane, and* $\operatorname{Im}(\phi)$ *is invariant under the action on* $\mathbb{R}^{1+2}$ *by the group of isometries generated by the "corkscrew" motion* $Q(x^0, x^1, x^2) = (x^0+\lambda, -x^1+1, -x^2)$ *(*Q *is a combination of a translation of* $\mathbb{R}^{1+2}$ *in time (i.e.* $(x^0, x^1, x^2) \mapsto (x^0+\lambda, x^1, x^2)$*) and a spatial rotation of* $\mathbb{R}^{1+2}$ *by* π *radians leaving invariant the line* $\{(x^0, \frac{1}{2}, 0)\colon x^0 \in \mathbb{R}\}$ *(i.e.* $(x^0, x^1, x^2) \mapsto (x^0, -x^1+1, -x^2)$*) and so* Q *satisfies, in particular, the translation identity* $Q^2(x^0, x^1, x^2) = (x^0+2\lambda, x^1, x^2)$*).*

Proof. To analyse ϕ we will use the method of images. We introduce a C^2 proper immersion $\hat{c}\colon \mathbb{R}\to\mathbb{R}^2$, which extends c periodically, defined as follows. We define $\hat{c}$ on $[-\lambda, \lambda]$ by

$$\hat{c}(s) = \begin{cases} c(s) & \text{for } s \in [0,\lambda] \\ -c(-s) & \text{for } s \in [-\lambda, 0] \end{cases}$$

and we then extend $\hat{c}$ to all of $\mathbb{R}$ periodically by

$$\hat{c}(s \pm 2\lambda) = \hat{c}(s) \pm (2, 0).$$

Next we introduce a vector field $\hat{v}$ along $\hat{c}$. We define $\hat{v}$ on $[-\lambda, \lambda]$ by

$$\hat{v}(s) = \begin{cases} v(s) & \text{for } s \in [0, \lambda] \\ -v(-s) & \text{for } s \in [-\lambda, 0] \end{cases}$$

and we then extend $\hat{v}$ to all of $\mathbb{R}$ periodically by

$$\hat{v}(s) = \hat{v}(s \pm 2\lambda).$$

The pair $(\hat{c}, \hat{v})$ thus satisfy

$$\hat{c}'(s) = \hat{c}'(-s) \tag{6.18}$$

$$\hat{c}'(s + 2\lambda) = \hat{c}'(s) \tag{6.19}$$

$$\hat{v}(s) = -\hat{v}(-s) \tag{6.20}$$

$$\hat{v}(s + 2\lambda) = \hat{v}(s) \tag{6.21}$$

for all $s \in \mathbb{R}$. Note that it follows from (6.18)–(6.21) together with the compatibility conditions (6.13)–(6.17) that the pair $(\hat{c}, \hat{v})$ is $C^2 \times C^1$. Define a C^2 map $\hat{\phi} \colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ by

$$\hat{\phi}(s, t) = (t, \hat{\gamma}(s, t))$$

where $\hat{\gamma} \colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the initial value problem

$$\begin{aligned} \hat{\gamma}_{tt} - \hat{\gamma}_{ss} &= 0 \\ \hat{\gamma}(s, 0) &= \hat{c}(s) \\ \hat{\gamma}_t(s, 0) &= \hat{v}(s). \end{aligned}$$

Note that $\hat{\gamma}$ is given by the d'Alembert formula

$$\hat{\gamma}(s,t) = \frac{1}{2}\left(\hat{c}(s+t) + \hat{c}(s-t) + \int_{s-t}^{s+t} \hat{v}(\zeta)d\zeta\right). \tag{6.22}$$

From (6.18)–(6.21) and (6.22) we may compute that

$$\hat{\gamma}_t(0,t) = \hat{\gamma}_t(\lambda,t) = 0.$$

Thus by uniqueness of solutions to (6.16) we have

$$\phi(s,t) = \hat{\phi}(s,t) \tag{6.23}$$

for $(s,t) \in [0,\lambda] \times [0,\infty)$.

Note that from (6.18)–(6.21) and (6.22) it follows that ϕ is an immersion iff $\hat{\phi}$ is an immersion. In fact, from (6.18)–(6.21) one may derive the identities

$$\hat{\phi}(s+2\lambda,t) = S\Big(\hat{\phi}(s,t)\Big) \tag{6.24}$$

$$\hat{\phi}(-s,t) = R\Big(\phi(s,t)\Big) \tag{6.25}$$

$$\hat{\phi}(s,t+\lambda) = \hat{\phi}(s+\lambda,t) + (\lambda,-1,0) = Q\Big(\hat{\phi}(\lambda-s,t)\Big) \tag{6.26}$$

for all $(s,t) \in \mathbb{R}^2$ where $S(x^0,x^1,x^2) = (x^0,x^1+2,x^2)$ denotes a translation in space, $R(x^0,x^1,x^2) = (x^0,-x^1,-x^2)$ denotes a rotation about the x^0 axis (leaving the x^1–x^2 plane invariant) of π radians, and $Q(x^0,x^1,x^2) := R(x^0,x^1,x^2)+(\lambda,1,0) = (x^0+\lambda,-x^1+1,-x^2)$ is a combination of a translation and a rotation such that $Q^2(x^0,x^1,x^2) = (x^0+2\lambda,x^1,x^2)$ is a translation forward in time. Thus $\mathrm{Im}(\hat{\phi})$ is invariant under the action of the group G of isometries on $\mathbb{R}^{1+2}$ which is generated by S, R and Q and, moreover, if $\hat{\phi}$ is an immersion then the action on $\mathbb{R}^{1+2}$ by any element of G carries $\mathrm{Im}(\phi)$ isometrically onto a subset of $\mathrm{Im}(\hat{\phi})$.

Let us now observe that $\hat{\phi}$ is an immersion iff there exist a pair of disjoint open semi-circles $\Lambda_+ \subseteq S^1$ and $\Lambda_- \subseteq S^1$ such that $\mathrm{Im}(a_+) \subseteq \Lambda_+$ and $\mathrm{Im}(a_-) \subseteq \Lambda_-$. Indeed, writing $\hat{a}_\pm(s) = \hat{v}(s) \pm \dot{\hat{c}}(s)$ we have $\hat{\gamma}_s(s,t) = \frac{1}{2}(\hat{a}_+(s+t) - \hat{a}_-(s-t))$ so $\hat{\phi}$ is an immersion iff $\hat{a}_+(\xi) \neq \hat{a}_-(\eta)$ for all $\xi, \eta \in \mathbb{R}$ which from (6.18)–(6.21) is the case iff

$$a_+(\xi) \neq a_-(\eta)$$
$$a_+(\xi) \neq -a_+(\eta)$$
$$a_-(\xi) \neq -a_-(\eta)$$

i.e. iff $\mathrm{Im}(a_+)$ and $\mathrm{Im}(a_-)$ are contained in disjoint semi-circles.

Now let us show that if ϕ is an immersion then ϕ is an embedding and $\mathrm{Im}(\phi)$ is a C^2 graph over some timelike plane. If ϕ is an immersion then $\hat{\phi}$ is an immersion and so $\mathrm{Im}(\hat{a}_+)$ and $\mathrm{Im}(\hat{a}_-)$ are contained in disjoint semi-circles. Since $\hat{a}_\pm \colon \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$ are periodic, by Lemma 2.6 there then exists $\omega \in \mathbb{R}^2$ such that

$$\langle \hat{\gamma}_s(s,t), \omega \rangle = \frac{1}{2} \langle \hat{a}_+(s+t) - \hat{a}_-(s-t), \omega \rangle > 0 \tag{6.27}$$

for all $(s,t) \in \mathbb{R} \times [0, \infty)$ and it follows (refer to the proof of Theorem 2.3) that $\hat{\phi}$ is an embedding and $\mathrm{Im}(\hat{\phi})$ is a C^2 graph over the timelike plane $P = \mathrm{span}\Big\{(1,0,0), (0,\omega)\Big\}$, so certainly ϕ is an embedding and $\mathrm{Im}(\phi)$ is a graph over P.

Finally, from (6.26) we have $\phi(s,t) = Q(\phi(\lambda - s, t))$ for all $(s,t) \in [0,\lambda] \times \mathbb{R}$ where $Q(x^0, x^1, x^2) = (x^0 + \lambda, -x^1 + 1, -x^2)$, so $\mathrm{Im}(\phi)$ is invariant under the action on $\mathbb{R}^{1+2}$ by Q. This completes the proof. $\square$

Remark 6.12 (An alternative method for small data)**.** If the initial data (C, V) is suitably small, then it is possible to prove that the evolution by isothermal gauge $\phi \colon [0,\lambda] \times \mathbb{R} \to \mathbb{R}^{1+2}$ is an immersion and $\mathrm{Im}(\phi)$ is a C^2 graph without using the method of images (i.e. without relying on an explicit representation formula) by using energy conservation instead. To see this, let the initial data (C, V) be parametrized isothermally so that $C(s) = (0, c(s))$, $V(s) = (1, v(s))$ where $\langle c'(s), v(s)\rangle = 0$ and $|c'(s)|^2 + |v(s)|^2 = 1$, and suppose that one has the bound

$$\lambda \left(\int_0^\lambda |c''(s)|^2 + |v'(s)|^2 ds \right) \leq \frac{1}{5}. \tag{6.28}$$

Note that the left hand side of (6.28) is invariant under the rescaling $(x^0, x^1, x^2) \mapsto (\mu x^0, \mu x^1, \mu x^2)$ of $\mathbb{R}^{1+2}$. In the special case $v \equiv 0$, we have that the parameter $s = l$ is the arclength of C, $\lambda = L$ is the length of C, and $|c''(l)| = |k(l)|$ is the absolute value of the curvature of C, and the bound (6.28) reads

$$L \left(\int_0^L |k(l)|^2 dl \right) \leq \frac{1}{5}.$$

We will now prove using energy conservation that (6.28) implies ϕ is an immersion and $\mathrm{Im}(\phi)$ is a C^2 graph. Let $\phi \colon [0, \lambda] \times \mathbb{R} \to \mathbb{R}^{1+2}$, $\phi(s, t) = (t, \gamma(s, t))$ be the evolution of $(C, V, \Gamma_1, \Gamma_2)$ by isothermal gauge as in Definition 6.10, and we are to repeat the proof that ϕ is an immersion and $\mathrm{Im}(\phi)$ is a C^2 graph by energy conservation. Note that ϕ is an immersion provided $|\gamma_s(s, t)|^2 > 0$ for all $(s, t) \in [0, \lambda] \times \mathbb{R}$, which since $|\gamma_s|^2 + |\gamma_t|^2 = 1$ is equivalent to

$$|\gamma_t(s, t)|^2 < 1. \tag{6.29}$$

Since

$$\gamma_t(0, t) = \gamma_t(\lambda, t) = 0 \tag{6.30}$$

by Cauchy-Schwarz we have

$$\begin{aligned}
|\gamma_t(s,t)|^2 &= \left(\int_0^t \frac{\partial}{\partial s}|\gamma_t(s,t)|ds\right)^2 \\
&\leq \left(\int_0^\lambda |\gamma_{st}(s,t)|ds\right)^2 \\
&\leq \lambda\left(\int_0^\lambda |\gamma_{st}(s,t)|^2 ds\right).
\end{aligned} \tag{6.31}$$

Defining

$$E(t) = \int_0^\lambda |\gamma_{ss}(s,t)|^2 + |\gamma_{st}(s,t)|^2 ds \tag{6.32}$$

by (6.30) we have

$$\begin{aligned}
\dot{E}(t) &= 2\int_0^\lambda \langle\gamma_{ss}(s,t),\gamma_{sst}(s,t)\rangle + \langle\gamma_{st}(s,t),\gamma_{stt}(s,t)\rangle ds \\
&= 2\int_0^\lambda \frac{\partial}{\partial s}\langle\gamma_{ss}(s,t),\gamma_{st}(s,t)\rangle + \langle-\gamma_{sss}(s,t)+\gamma_{stt}(s,t),\gamma_{st}(s,t)\rangle ds \\
&= 2\int_0^\lambda \frac{\partial}{\partial s}\langle\gamma_{ss}(s,t),\gamma_{st}(s,t)\rangle ds \\
&= 2\left(\langle\gamma_{ss}(\lambda,t),\gamma_{st}(\lambda,t)\rangle - \langle\gamma_{ss}(0,t),\gamma_{st}(0,t)\rangle\right) \\
&= 2\left(\langle\gamma_{tt}(\lambda,t),\gamma_{st}(\lambda,t)\rangle - \langle\gamma_{tt}(0,t),\gamma_{st}(0,t)\rangle\right) \\
&= 0
\end{aligned} \tag{6.33}$$

so

$$E(t) = E(0) \tag{6.34}$$

for all $t \in \mathbb{R}$.[2] Putting together (6.31), (6.34) and (6.28) gives

$$|\gamma_t(s,t)|^2 \leq \lambda E(t) = \lambda E(0) \leq \frac{1}{5}. \tag{6.35}$$

So we have established (6.29) which proves that ϕ is an immersion. Now we will show that $\mathrm{Im}(\phi)$ is a C^2 graph. Writing

$$U(s,t) = \frac{\gamma_s(s,t)}{|\gamma_s(s,t)|} \tag{6.36}$$

for the spatial unit tangent along ϕ, we will show that $\mathrm{Im}(U) \subseteq S^1$ is contained in an open semi-circle, which will imply that $\mathrm{Im}(\phi)$ is a C^2 graph. To show this, observe that by (6.35) we have

$$|\gamma_s(s,t)|^2 = 1 - |\gamma_t(s,t)|^2 \geq \frac{4}{5}$$

so

$$\left|\frac{\partial U}{\partial s}(s,t)\right| = \left|\frac{\gamma_{ss}(s,t)}{|\gamma_s(s,t)|} - \frac{\langle \gamma_{ss}(s,t), \gamma_s(s,t)\rangle}{|\gamma_s(s,t)|^3}\gamma_s(s,t)\right| \leq \sqrt{5}|\gamma_{ss}(s,t)|. \tag{6.37}$$

Since $\gamma_t(0,t) = 0$ implies

$$U(0,t) = U(0,0)$$

[2] In fact a few words are needed to justify the computation (6.33). Since γ is only C^2, a-priori we do not have the regularity required to carry out the computation (6.33), and (6.33) should only be considered formal. Nonetheless, the desired energy conservation (6.34) holds for C^2 solutions to the wave equation as may be checked, for example, directly from a d'Alembert formula.

for all $t \in [0, \infty)$, by Cauchy-Schwarz, (6.37), (6.34) and (6.28) we arrive at

$$
\begin{aligned}
|U(s,t) - U(0,0)|^2 = |U(s,t) - U(0,t)|^2 &= \left| \int_0^s \frac{\partial U}{\partial s}(s,t)ds \right|^2 \\
&\leq \lambda \left(\int_0^\lambda \left| \frac{\partial U}{\partial s}(s,t) \right|^2 ds \right) \leq 5\lambda \left(\int_0^\lambda |\gamma_{ss}(s,t)|^2 ds \right) \\
&\leq 5\lambda E(t) = 5\lambda E(0) \leq 1
\end{aligned}
\tag{6.38}
$$

for all $(s,t) \in [0, \lambda] \times [0, \infty)$, which implies that $\mathrm{Im}(U)$ is contained in an open semi-circle as claimed.

Chapter 7

Future questions

We have now given the proofs of all of the theorems in this thesis, and so we have come to the end of our story. In this final chapter, we will outline (in no particular order) some interesting questions for the future.

It follows from Theorem 2.3 that if $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ is a smooth proper timelike maximal immersion and $N\colon \mathbb{R}^2 \to S^{1+1}$ is the (spacelike) unit normal along ϕ, then for every compact subset $K \subseteq \mathbb{R}^2$, $N(K)$ is contained in an open hemi-hyperboloid. In particular, it follows that $\mathrm{Im}(N)$ is contained in a closed hemi-hyperboloid. Example 2.9 shows a situation where the closure of $\mathrm{Im}(N)$ in S^{1+1} intersects both connected components of the boundary of a closed hemi-hyperboloid, but we don't have any examples where $\mathrm{Im}(N)$ is dense in a closed hemi-hyperboloid.

Problem 7.1 (Bounds on the unit normal)**.** *For an arbitrary smooth properly immersed timelike maximal surface in $\mathbb{R}^{1+2}$ with spacelike unit normal N, what (in some appropriate sense) is the sharpest possible bound on the size of $\mathrm{Im}(N)$?*

Next, a natural question concerns the lowest regularity required for Theorem 2.3. Specifically, what is the smallest $k \geq 1$ (including non-integers) such that every C^k proper timelike maximal immersion $\phi\colon \mathbb{R}^2 \to \mathbb{R}^{1+2}$ (see Appendix B for definition) is an embedding whose image is a C^k graph over compact subsets? An examination of our proof of Theorem 2.3 shows that this is ultimately related to the following problem.

Problem 7.2 (The conformal Bernstein problem in low regularity). *What is the smallest* $k \geq 1$ *(including non-integers) such that every* C^k *properly immersed timelike maximal surface in* $\mathbb{R}^{1+2}$ *is* C^1 *conformally equivalent to the Minkowski plane* $\mathbb{R}^{1+1}$*?*

The statement "*there exists no smooth proper timelike maximal immersion* $\phi\colon S^1 \times \mathbb{R} \to M^{1+2}$" is true if $M^{1+2} = \mathbb{R}^{1+2}$ (by [59, Theorem 1.1]) but is false for a general globally hyperbolic Lorentzian manifold M^{1+2}. Indeed, if $M^{1+2} = \mathbb{R} \times \Sigma^2$ is equipped with the static metric $-dt^2 + g_{\Sigma^2}$ where g_{Σ^2} is a complete Riemannian metric on the surface Σ^2 and if $c\colon S^1 \to \Sigma^2$ is a closed geodesic, then the immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R} \times \Sigma^2$ defined by $\phi(s,t) = (t, c(s))$ gives a smooth proper timelike maximal immersion (the Cauchy evolution of the initial data set (γ, ∂_t)). In the case that $\Sigma^2 = S^2$ with the round metric g_{S^2} and $c\colon S^1 \to \Sigma^2$ traces a great circle, it may be shown that the resulting timelike maximal surface in $\mathbb{R} \times S^2$ is unstable, in the sense that there exist smooth initial data sets (C, V) lying arbitrarily close to the initial data (c, ∂_t) for which the Cauchy evolution of (C, V) to a timelike maximal surface in $\mathbb{R} \times S^2$ becomes singular in finite time. It seems natural to interpret this as a consequence of the fact that the great circles are unstable critical points for the length functional on S^2.

Problem 7.3 (Stability of static timelike maximal surfaces in static spacetimes). *If* $c\colon S^1 \to \Sigma^2$ *is a closed geodesic with respect to a complete Riemannian metric* g_{Σ^2} *on* Σ^2 *which is a strict local minimizer of the length functional, is the corresponding Cauchy evolution for a static timelike maximal surface in the static spacetime* $(\mathbb{R} \times \Sigma^2, -dt^2 + g_{\Sigma^2})$ *stable with respect to small perturbations of the initial data* (c, ∂_t)*?*

The only case of the IBVP for a pair of timelike boundary curves (see the first paragraph of §6.2 for precise statement of this IBVP) that we considered in this thesis was the case of a pair of parallel timelike lines (Theorem 6.11) as this is the only case in which we can obtain an explicit global representation formula for conformally parameterised solutions. For the general case of this IBVP, the representation formulas one obtains by conformal methods seem to become far to complicated to analyse and conformal methods are possibly not the way forward here. One case which is of interest, however, which might be partially tractable by conformal methods, is the case of a pair of non-parallel timelike boundary lines. Note that in this case one at least has an explicit representation formula for the solution locally (see §5.7 and exploit finite speed of propagation for wave equations).

Problem 7.4 (IBVP for non-parallel lines)**.** *For the IBVP for a timelike maximal surface in $\mathbb{R}^{1+2}$ with timelike boundary consisting of a pair of non-parallel and non-intersecting timelike straight lines, do there exist any global solutions? If so, what are their stability properties?*

Recall that, at least generically, the method of isothermal gauge does not give a good notion of solution after singularity has formed (it forms a swallowtail which is not even a C^1 surface). On the other hand, we know that, for generic smooth initial data, the limit curve at the first time of singularity will be C^1 (and the surface will become null at this point). This begs the question as to whether, in the generic case, there might exist (even locally) an extension of the surface beyond the first time of singularity to a C^1 causal surface which is a timelike maximal surface away from some singular set.

Problem 7.5 (C^1 causal extendibility)**.** *Suppose that one is given a C^1 causal immersion $\phi\colon \Omega_\varepsilon \to \mathbb{R}^{1+2}$ where $\Omega_\varepsilon := (-\varepsilon, \varepsilon) \times (-\varepsilon, 0]$ of the form $\phi(s,t) = (t, \gamma(s,t))$ which is a smooth timelike maximal immersion on $\Omega_\varepsilon \setminus (0,0)$ and which is null at* *the point* $(0,0)$*. By redefining* $\varepsilon > 0$ *to be smaller if necessary (i.e. by redefining the domain of* ϕ *to be the subset* $\Omega_{\varepsilon'} \subset \Omega_\varepsilon$ *for some* $0 < \varepsilon' < \varepsilon$ *if necessary), can one construct an extension* $\tilde{\phi}\colon U \to \mathbb{R}^{1+2}$ *of* ϕ *to an open subset* $U \subseteq \mathbb{R}^2$ *containing* Ω_ε *which is a* C^1 *causal immersion and which is a timelike maximal immersion on* $U \setminus \mathcal{K}$ *where* $\mathcal{K} \subseteq U$ *is a subset of Hausdorff dimension* ≤ 1*?*

Many of the methods that we have employed in this thesis (specifically, the conformal methods) will not in general be applicable to timelike maximal surfaces in higher dimensions. Nonetheless, there are many natural and interesting questions in higher dimensions. Theorem 2.3 will not generalize to higher dimensions, in the sense that there exist smooth proper timelike maximal immersions $\phi\colon \mathbb{R}^3 \to \mathbb{R}^{1+3}$ which are not embeddings (indeed, for an example, let $\gamma\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterisation of Enneper's self-intersecting minimal surface (see Figure 1.1) and define $\phi\colon \mathbb{R}^3 \to \mathbb{R}^{1+3}$ by $\phi(x,t) = (t, \gamma(x))$, then ϕ is a smooth proper self-intersecting timelike maximal immersion[1]). On the other hand, Wong [75] has observed that there are currently no counter-examples to the statement "*for any compact manifold* M^{n-1} *there exists no smooth proper timelike maximal immersion* $\phi\colon M^{n-1} \times \mathbb{R} \to \mathbb{R}^{1+n}$" and described this statement as "tempting" to conjecture. One small step towards such a conjecture might be to understand in better detail the stability of "shrinking sphere" solutions $\phi\colon S^{n-1} \times (-T_*, T_*) \to \mathbb{R}^{1+n}$ which are higher dimensional analogues of the shrinking circle solution of Example 1.10. A related problem is the stability of the identity wave map between two spheres, which is defined by $\phi\colon S^m \times \mathbb{R} \to S^m$; $\phi(p,t) = p$. For the case $m \geq 3$, one expects instability of this wave map, since the identity harmonic map $\mathrm{Id}\colon S^m \to S^m$ is unstable. For the case $m = 1$, the wave map equation may be solved explicitely, and it is an interesting exercise to observe that the identity wave map is stable in the sense that small perturbations of the identity initial data lead

[1] note that the stability of such self-intersecting timelike maximal surfaces, on the other hand, may be another matter

to a globally regular wave map which stays close to some member of the 1-parameter family of rotating solutions $\{\phi_\beta\colon S^1 \times \mathbb{R} \to S^1\}_{\beta\in\mathbb{R}}$ given by $\phi_\beta(e^{i\theta}, t) = e^{i(\theta+\beta t)}$. This leaves open an interesting borderline case when $m = 2$.

Problem 7.6 (A problem concerning wave maps)**.** *What are the stability properties of the identity wave map $\phi\colon S^2 \times \mathbb{R} \to S^2$ defined by $\phi(p,t) = p$?*

Theorem 3.1 gives us a description of singularity formation in terms of a spatial curvature blow-up for (1+1)-dimensional timelike maximal surfaces in $\mathbb{R}^{1+2}$, with a curvature blow-up in $L^1_{time}L^\infty_{space}$-norm. Recall that in Example 3.3 we computed the rate of curvature blow-up in more detail for the special case of the shrinking circle, and we observed that there is also blow-up in $L^p_{time}L^q_{space}$ norm whenever $p, q \in (0, \infty)$ are Hölder conjugate i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 7.7 (Curvature blow-up in Hölder conjugate norms)**.** *Does singularity formation for timelike maximal surfaces in $\mathbb{R}^{1+2}$ always involve a blow-up of spatial curvature in $L^p_{time}L^q_{space}$-norm for all Hölder conjugate $p, q \in (0, \infty)$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$)?*

More generally/ambitiously, there is still lots to be understood regarding singularity formations in higher dimensions and codimensions

Problem 7.8 (Blow-up in higher dimensions)**.** *What are the mechanisms of curvature blow-up during singularity formation for timelike maximal surfaces in higher dimensions and codimensions?*

Finally, often in this thesis we have made use of the word "generic", without providing any rigorous definition. There appears to be some scope for trying to make precise what exactly is meant by generic in our context, and a better understanding of the term might give insights into problems involving timelike maximal surfaces in higher dimensions. Let us discuss just one instance in which the word generic occurs in the context of timelike maximal surfaces. Recall that it was proved by Jerrard, Novaga & Orlandi [42] that for generic smooth initial data (C, V) where $C\colon S^1 \to \mathbb{R}^4$, the evolution of (C, V) by isothermal gauge yields a smooth proper timelike maximal immersion $\phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^{1+4}$. The authors in [42] showed that this result may effectively be deduced from the following statement:

$$\text{“}\textit{given a pair of smooth curves } \gamma_1\colon S^1 \to S^3 \textit{ and } \gamma_2\colon S^1 \to S^3, \textit{ generically, } \gamma_1 \textit{ and } \gamma_2 \textit{ will not intersect”.} \tag{$\star$}$$

Visualizing S^3 as the one-point compactification of $\mathbb{R}^3$, and the statement $(\star)$ may seem intuitively obvious. But it is an interesting problem to try and make $(\star)$ precise. In [42], the authors prove that the space of pairs of closed curves in S^3 may be equipped with a natural topology with respect to which the subset K of pairs of closed curves which intersect forms a closed subset of empty interior. But there seems to be room for more to be said (indeed, think of the fat Cantor set in the unit interval $[0, 1]$, which is also a closed subset of empty interior). It would be interesting to understand what are the possible measures on the space of curves with respect to which K has measure zero.

Problem 7.9 (Making genericity precise)**.** *What is a good measure on the space of pairs of closed curves in S^3 with respect to which the statement $(\star)$ may be made precise?*

Appendix A

Roadmap (structure of the thesis)

Structure of Chapter 1. Chapter 1 contains some context to the topic of timelike maximal surfaces. This starts in §1.1.1 with a rapid history of minimal surfaces in $\mathbb{R}^3$, which are the better-known cousins of timelike maximal surfaces in $\mathbb{R}^{1+2}$. In §1.1.2 we briefly discuss Einstein's 'beautiful thought'—since an important motivation for Lorentzian geometry comes from the theory of relativity—and in §1.1.3 we introduce timelike maximal surfaces alongside some relevant physical theories. In §1.2.1 an appropriate notion of global solution is defined and we introduce the initial value problem. In §1.2.3 we review the method of isothermal gauge, which is a method in a sense analogous to the Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$, and we see some well-known examples of timelike maximal surfaces: the timelike plane and the shrinking circle. In §1.3.1 we review some singularity results for timelike maximal surfaces in $\mathbb{R}^{1+2}$ and in §1.3.2–1.3.3 we discuss some further relevant literature including stability results in all dimensions and codimensions. In §1.4 we state the main results of this thesis.

Structure of Chapters 2–4. In §2.1 we derive a smooth conformal equivalence between the induced metric on an arbitrary smooth properly immersed timelike maximal surface and the Minkowski plane $\mathbb{R}^{1+1}$ (Lemma 2.2). In §2.2 we prove Theorem

2.3 (on embeddedness of timelike maximal surfaces) and in §2.3 we see examples of both graphical and non-graphical timelike maximal surfaces. The latter examples show that the restriction to compact subsets in Theorem 2.3 cannot be relaxed in general. In §3.1 we prove Theorem 3.1 (on singularity formation) and in §3.2 we discuss in a bit more detail the rate of curvature blow-up for the special case of the shrinking circle (Example 3.3 and Remarks 3.4–3.5). Chapter 4 is devoted to analysis in isothermal gauge. In §4.1 we define the evolution by isothermal gauge for a $C^1 \times C^0$ initial data and gather some basic results. In §4.2 we further analyse the solution by isothermal gauge. In particular we prove a localized singularity statement to complement Theorem 2.3 (Proposition 4.5) and we observe local and global existence results which are notable in that they require no decay on the initial data at infinity (Corollary 4.13 and Remark 4.10). In §4.3 we see examples illustrating some non-generic singular behaviours, including C^1 properly embedded surfaces containing non-graphical compact sets which are smooth timelike maximal surfaces away from a pair of null half-lines (Example 4.14) and C^1 properly embedded graphical, but not C^1 graphical, periodic surfaces which are smooth timelike maximal surfaces away from a discrete lattice of null points (Example 4.15). In §4.4 we prove Theorem 4.16 (on global C^1 inextendibility in isothermal gauge) and we see some more examples of possible non-generic singular behaviours (Examples 4.19 and 4.20). In §4.5 we observe a non-generic example in which singularity formation is not by collapse but for which the limit curve at the first singularity is not C^1.

Structure of Chapters 5 & 6. In Chapters 5 & 6 we move on to IBVPs for timelike maximal surfaces in $\mathbb{R}^{1+2}$. Chapter 5 treats in detail the situation for a single prescribed timelike boundary curve. In §5.1 we give the precise statement of this IBVP and we introduce natural C^2 compatibility conditions on the initial-boundary data which are necessary for the existence of a C^2 solution to the problem.

In §5.2 we make a choice of conformal structure for the solution of the IBVP (see Figure 5.2) and we see that this imposes (unnatural) C^2 compatibility conditions on the data 'at the corner'. In §5.3 we set about solving the IBVP. To be precise, we define a notion of evolution by isothermal gauge (i.e. Weierstrass-type formula, see Definition 5.4) and we prove that this gives a C^2 solution to the IBVP modulo possible singular points (Proposition 5.5). In §5.4 we analyse the singular points, and in particular we write down a condition which implies that singular points correspond to curvature blow-ups (Lemma 5.8). In §5.5 is a quick example exhibiting solutions of the IBVP with each of the two distinct structures of null infinity of our conformal domain. In §5.6 we give a sufficient condition on the initial-boundary data for the singular set to be non-empty (Proposition 5.12) and so we arrive at a non-empty open set of initial-boundary data for which we have global solutions to the IBVP (our choice of the conformal structure of null infinity for the solution thus turns out to be justified a postiori). In Remark 5.14 we loosely discuss the "size" of this open set and we note, in particular, that our results imply a C^1 stability result for the quadrant of a timelike plane with respect to this IBVP (Remark 5.13). In §5.7 we treat the special case that the timelike boundary is a half-line. In this case our equations are integrable, we obtain an explicit expression for the evolution by isothermal gauge, and we can completely classify the initial data sets in terms of singularity vs. no singularity (Theorem 5.15). We then illustrate in this special case a situation in which singularity forms only outside of the domain of dependence of the initial curve i.e. singularity forms due to the 'reflection of waves off the boundary' (Example 5.17). In Chapter 6, as an application of the results of Chapter 5, we treat two further IBVPs with timelike boundaries consisting respectively of: (i) a single timelike plane, and (ii) a pair of parallel timelike lines (with general initial data considered in both cases). The case (i) is treated in §6.1 where a method is developed which could in principle be applied to treat any timelike boundary surface. We again derive C^2 compatibility conditions here which follow from the choice of conformal structure of the corner, but we observe that only one type of null infinity is possible in this setting (Remark 6.2). We see that our equations are integrable in the case of the timelike plane and we can classify the space of solutions by isothermal gauge (Theorem 6.6). The case (ii) is treated in §6.2 by the method of images, yielding an explicit representation formula with which we classify the space of solutions by isothermal gauge (Theorem 6.11), and also (with less optimal results) by energy methods (Remark 6.12).

Appendix B

Notation & terminology

We work throughout the thesis in Minkowski space $\mathbb{R}^{1+2}$. Let (x^0, x^1, x^2) denote coordinates on $\mathbb{R}^{1+2}$ such that the Minkowski metric is

$$\eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2.$$

We call such coordinates (x^0, x^1, x^2) inertial.

For $V \in T_p\mathbb{R}^{1+2}$ we write

$$V = (V^0, V^1, V^2) = (V^0, v^1, v^2) = (V^0, v),$$

i.e. we always use an uppercase letter for a vector and a lowercase letter for the spatial component of a vector (with respect to some fixed inertial coordinates on $\mathbb{R}^{1+2}$).

For $V, W \in T_p\mathbb{R}^{1+2}$ it is convenient to introduce the shorthand

$$\begin{aligned} \langle\!\langle V, W \rangle\!\rangle &= -V^0W^0 + v^1w^1 + v^2w^2 = -V^0W^0 + \langle v, w\rangle \\ \|V\|^2 &= \langle\!\langle V, V \rangle\!\rangle = -(V^0)^2 + |v|^2 \end{aligned}$$

for the Minkowskian inner product, where we make the standard abuse of notation in identifying $T_p\mathbb{R}^{1+2} = \mathbb{R}^{1+2}$ for all $p \in \mathbb{R}^{1+2}$. We reserve the notation $\langle V, W\rangle = \sum_i V^i W^i$, and $|V|^2 = \langle V, V\rangle$ for the usual Euclidean inner product.

A vector $V \in T_p\mathbb{R}^{1+2}$ is timelike if $\|V\|^2 < 0$, spacelike if $\|V\|^2 > 0$, null if $\|V\|^2 = 0$ and causal if it is either timelike or null. Note that by this terminology a timelike or spacelike vector field is necessarily nowhere-vanishing. A timelike or null vector V is called future-directed (resp. past-directed) if $V^0 > 0$ (resp. $V^0 < 0$).

Let $\Omega \subseteq \mathbb{R}^2$ be an open subset and $\phi\colon \Omega \to \mathbb{R}^{1+2}$ be a C^1 immersion. Let (s,t) denote coordinates on $\Omega \subseteq \mathbb{R}^2$ and write $\phi^\alpha = x^\alpha \circ \phi$ for the expression of ϕ in inertial coordinates. The induced metric $g = \phi^*\eta$ (or first fundamental form) is the bilinear form

$$g(s,t) = E(s,t)ds^2 + 2F(s,t)dsdt + G(s,t)dt^2$$

where

$$E(s,t) = \|\phi_s(s,t)\|^2, \quad F(s,t) = \langle\!\langle \phi_s(s,t), \phi_t(s,t)\rangle\!\rangle, \quad G(s,t) = \|\phi_t(s,t)\|^2.$$

For each $q \in \Omega$ we say that ϕ is timelike at q if $\det(g(q)) < 0$, ϕ is null at q if $\det(g(q)) = 0$, ϕ is spacelike at q if $\det(g(q)) > 0$, and ϕ is causal at q if ϕ is either timelike or null at q. We say that ϕ is timelike (resp. causal) if it is timelike (resp. causal) at every point $q \in \Omega$. In the case that ϕ is timelike at q there exists a choice of unit normal vector $N(q)$ (which is spacelike) and a direct sum decomposition of the tangent space which is orthogonal with respect to $\eta = \langle\!\langle \cdot, \cdot \rangle\!\rangle$

$$T_{\phi(q)}\mathbb{R}^{1+2} = \operatorname{span}\{N(q)\} \oplus \operatorname{Im}(d\phi_q).$$

Thus if $\phi\colon \Omega \to \mathbb{R}^{1+2}$ is a C^2 timelike immersion then one may define the second fundamental form

$$\mathrm{II}(s,t) = e(s,t)ds^2 + 2f(s,t)dsdt + g(s,t)dt^2$$

and mean curvature in the same way as in the familiar Euclidean setting, see e.g. [70, Chap. 7].

On any smooth properly immersed timelike surface $\Sigma \subseteq \mathbb{R}^{1+2}$ there exist a pair of smooth future-directed null tangent vector fields which provide a global frame for the tangent bundle (i.e. they span the tangent space at every point). We may (arbitrarily) assign the names incoming and outgoing to these vector fields respectively. Locally about any point on the surface, the integral curves of the incoming and outgoing null vector fields may be chosen to serve as coordinate lines. We call a C^1 local system of coordinates (z_+, z_-) on a C^1 timelike surface Σ null coordinates if $\frac{\partial}{\partial z_\pm}$ are future-directed null vector fields on Σ. If (z_+, z_-) are null coordinates then we call the coordinates (s,t) defined by $s = z_+ - z_-, t = z_+ + z_-$ isothermal coordinates. The induced metric in null or isothermal coordinates reads

$$g(z_+, z_-) = 2F(z_+, z_-)dz_+dz_- \quad \text{or} \quad g(s,t) = E(s,t)(ds^2 - dt^2).$$

If $\phi\colon \Omega \to \mathbb{R}^{1+2}$ is a C^1 timelike immersion then for every compact subset $U \subseteq \Omega$ the area of $\phi(U)$ is defined as

$$\mathcal{A}\left[\phi; U\right] = \int_U \sqrt{|\det(g(s,t))|}dsdt.$$

The area of $\phi(U)$ is independent of the choice of coordinates (s,t) on U. The Euler-

Lagrange equations associated to the area functional $\mathcal{A}$ are

$$\frac{1}{\sqrt{|\det g|}}\partial_i\left(\sqrt{|\det g|}g^{ij}\partial_j\phi^\alpha\right) = 0 \tag{B.1}$$

having adopted the summation convention. We call (B.1) the timelike maximal surface equation. We say that a C^1 timelike immersion $\phi\colon \Omega \to \mathbb{R}^{1+2}$ is a timelike maximal immersion if it satisfies (B.1) with respect to some coordinate system in the weak sense. We refer to the image of a timelike maximal immersion as a timelike maximal surface. When ϕ is a C^2 timelike immersion, the equation (B.1) is equivalent to $H(\phi) = 0$ where H denotes the mean-curvature vector.

The equation (B.1) is independent of the choice of coordinates in that if $\phi\colon \Omega \to \mathbb{R}^{1+2}$ is a C^2 solution to (B.1) and $\psi\colon \Omega' \to \Omega$ is a C^2 diffeomorphism, then $\phi' = \phi \circ \psi\colon \Omega' \to \mathbb{R}^{1+2}$ is also a C^2 solution to (B.1). The equation (B.1) is also invariant under the rescalings $(x^0, x^1, x^2) \mapsto (\lambda x^0, \lambda x^1, \lambda x^2)$ of $\mathbb{R}^{1+2}$ as well as the isometries of $\mathbb{R}^{1+2}$. For a C^1 timelike immersion, with respect to a system of null/isothermal coordinates the equation (B.1) reduces to the wave equation

$$\phi_{z_+z_-} = \phi_{tt} - \phi_{ss} = 0.$$

Bibliography

[1] P. Allen, L. Andersson, and J. Isenberg, *Timelike minimal submanifolds of general co-dimension in Minkowski space time*, J. Hyperbolic Differ. Equ. **3** (2006), no. 4, 691–700.

[2] J. Arnlind and J. Hoppe, *The world as quantized minimal surfaces*, Phys. Lett. B **723** (2013), no. 4-5, 397–400.

[3] V. I. Arnol'd, *Catastrophe Theory*, third ed., Springer-Verlag, Berlin, 1992.

[4] A. Aurilia and D. Christodoulou, *Theory of strings and membranes in an external field. I. General formulation*, J. Math. Phys. **20** (1979), no. 7, 1446–1452.

[5] H. Bahouri, A. Marachli, and G. Perelman, *Blow up dynamics for the hyperbolic vanishing mean curvature flow of surfaces asymptotic to Simons cone*, (2019), preprint, `https://arxiv.org/abs/1907.01126`.

[6] B. M. Barbashov and N. A. Chernikov, *Solution of the two plane wave scattering problem in a nonlinear scalar field theory of the Born-Infeld type*, J. Exptl. Theoret. Phys. **24** (1967), no. 2, 437–442.

[7] R. Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Comm. Math. Phys. **94** (1984), no. 2, 155–175.

[8] G. Bellettini, J. Hoppe, M. Novaga, and G. Orlandi, *Closure and convexity results for closed relativistic strings*, Complex Anal. Oper. Theory **4** (2010), no. 3, 473–496.

[9] G. Bellettini, M. Novaga, and G. Orlandi, *Time-like minimal submanifolds as singular limits of nonlinear wave equations*, Physica D: Nonlinear Phenomena **239** (2010), no. 6, 335–339.

[10] ______, *Lorentzian varifolds and applications to relativistic strings*, Indiana Univ. Math. J. **61** (2012), no. 6, 2251–2310.

[11] S. N. Bernstein, *Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique*, Comm. de la Soc. Math. de Kharkov (2éme sér.) **15** (1915–17), 38–45.

[12] D. Bonheure, F. Colasuonno, and J. Földes, *On the Born-Infeld equation for electrostatic fields with a superposition of point charges*, Ann. Mat. Pura Appl. (4) **198** (2019), no. 3, 749–772.

[13] M. Bordemann and J. Hoppe, *The dynamics of relativistic membranes. I. Reduction to 2-dimensional fluid dynamics*, Phys. Lett. **B317** (1993), no. 3, 315–320.

[14] ______, *The dynamics of relativistic membranes. II. Nonlinear waves and covariantly reduced membrane equations*, Phys. Lett. **B325** (1994), no. 3-4, 359–365.

[15] M. Born and L. Infeld., *Foundations of a new field theory*, Proc. R. Soc. Lond. **A144** (1934), no. 852, 425–451.

[16] S. Brendle, *Hypersurfaces in Minkowski space with vanishing mean curvature*, Comm. Pure Appl. Math. **55** (2002), no. 10, 1249–1279.

[17] Y. Brenier, *Non relativistic strings may be approximated by relativistic strings*, Methods Appl. Anal. **12** (2005), no. 2, 153–167.

[18] W. Busza, K. Rajagopal, and W. van der Sche, *Heavy ion collisions: The big picture, and the big questions*, Ann. Rev. Nucl. Part. Sci. **68** (2018), 339–376.

[19] E. Calabi, *Examples of Bernstein problems for some nonlinear equations*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 223–230.

[20] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, 1955.

[21] P. A. M. Dirac, *An extensible model of the electron*, Proc. R. Soc. Lond. **A268** (1962), no. 1332, 57–67.

[22] R. Donninger, J. Krieger, J. Szeftel, and W. W. Y. Wong, *Codimension one stability of the catenoid under the vanishing mean curvature flow in Minkowski space*, Duke Math. J. **165** (2016), no. 4, 723–791.

[23] J. Eggers and J. Hoppe, *Singularity formation for time-like extremal hypersurfaces*, Phys. Lett. **B680** (2009), no. 3, 274–278.

[24] J. Eggers, J. Hoppe, M. Hynek, and N. Suramlishvili, *Singularities of relativistic membranes*, Geom. Flows **1** (2015), no. 1, 17–33.

[25] J. Eggers and N. Suramlishvili, *Singularity theory of plane curves and its applications*, Eur. J. Mech. **B65** (2017), 107–131.

[26] A. Einstein, *Grundgedanken und Methoden der Relativitätstheorie in ihrer Entwicklung dargestellt: Berlin*, 1920.

[27] ———, *The Meaning of Relativity*, fifth ed., Princeton University Press, 1952.

[28] F. C. Eperon, H. S. Reall, and J. J. Sbierski, *Predictability of subluminal and superluminal wave equations*, Comm. Math. Phys. **368** (2019), no. 2, 585–626.

[29] S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada, and S.-D. Yang, *Zero mean curvature surfaces in Lorentz-Minkowski 3-space and 2-dimensional fluid mechanics*, Math. J. Okayama Univ. **57** (2015), 173–200.

[30] H. Fujimoto, *On the number of exceptional values of the Gauss maps of minimal surfaces*, J. Math. Soc. Japan **40** (1988), no. 2, 235–247.

[31] J. Gray, *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic*, Oxford University Press, 1979.

[32] C.-H. Gu, *On the Cauchy problem for harmonic maps defined on two-dimensional Minkowski space*, Comm. Pure Appl. Math. **33** (1980), no. 6, 727–737.

[33] ———, *On the motion of a string in a curved space-time*, Proceedings of the third Marcel Grossman meeting on general relativity (1983), 139–142.

[34] ———, *Complete extremal surfaces of mixed type in* 3*-dimensional Minkowski space*, Chinese Ann. Math. **B15** (1994), no. 4, 385–400.

[35] J. Hoppe, *Quantum theory of a massless relativistic surface and a two-dimensional bound state problem*, Ph.D. thesis, MIT, 1982.

[36] ———, *Conservation laws and formation of singularities in relativistic theories of extended objects*, (1995), preprint, `https://arxiv.org/abs/hep-th/9503069`.

[37] ———, *Curved space (matrix) membranes*, Gen. Relativity Gravitation **43** (2011), no. 9, 2523–2526.

[38] ———, *Relativistic membranes*, J. Phys. **A46** (2013), no. 2, 023001, 30.

[39] J. Hoppe and H. Nicolai, *Relativistic minimal surfaces*, Phys. Lett. **B196** (1987), no. 4, 451–455.

[40] J. Hoppe and M. Trzetrzelewski, *Lorentz-invariant membranes and finite matrix approximations*, Nuclear Phys. **B849** (2011), no. 3, 628–635.

[41] R. L. Jerrard, *Defects in semilinear wave equations and timelike minimal surfaces in Minkowski space*, Anal. PDE **4** (2011), no. 2, 285–340.

[42] R. L. Jerrard, M. Novaga, and G. Orlandi, *On the regularity of timelike extremal surfaces*, Commun. Contemp. Math. **17** (2015), no. 1, 1450048, 19.

[43] C. V. Johnson, *D-Branes*, Cambridge University Press, 2003.

[44] T. W. B. Kibble, *Topology of cosmic domains and strings*, J. Phys. **A9** (1976), no. 8, 1387–1398.

[45] D.-X. Kong, K. Liu, and Z.-G. Wang, *Hyperbolic mean curvature flow: evolution of plane curves*, Acta Math. Sci. (Engl. Ed.) **B29** (2009), no. 3, 493–514.

[46] R. S. Kulkarni, *An analogue of the Riemann mapping theorem for Lorentz metrics*, Proc. R. Soc. Lond. **A401** (1985), no. 1820, 117–130.

[47] J. L. Lagrange, *Essai d'une nouvelle methode pour determiner les maxima, et les minima des formules integrales indefinies*, Mélange de philosophie et de mathématique de la Société Royal de Turin (1760–1761), 173–195.

[48] B. Lambert, *Gradient estimates for spacelike mean curvature flow with boundary conditions*, Proc. Edinb. Math. Soc. **62** (2019), no. 2, 459–469.

[49] B. Lambert and J. Lotay, *Spacelike mean curvature flow*, (2018), to appear in J. Geom. Anal. (`https://arxiv.org/abs/1808.01994`).

[50] H. Lindblad, *A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time*, Proc. Amer. Math. Soc. **132** (2004), no. 4, 1095–1102.

[51] J. Liu and Y. Zhou, *Initial-boundary value problem for the equation of timelike extremal surfaces in Minkowski space*, J. Math. Phys. **49** (2008), no. 4, 043507, 26.

[52] ———, *The initial-boundary value problem on a strip for the equation of time-like extremal surfaces*, Discrete Contin. Dyn. Syst. **A23** (2009), no. 1-2, 381–397.

[53] G. K. Luli, S. Yang, and P. Yu, *On one-dimension semi-linear wave equations with null conditions*, Adv. Math. **329** (2018), 174–188.

[54] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Internat. J. Theoret. Phys. **38** (1999), no. 4, 1113–1133.

[55] J. B. Meusnier, *Mémoire sur la courbure des surfaces.*, Mém. Mathém. Phys. Acad. Sci. Paris, prés. par div. Savans **10** (1785), 477–510.

[56] T. K. Milnor, *A conformal analog of Bernstein's theorem for timelike surfaces in Minkowski* 3*-space*, The Legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985), Contemp. Math., vol. 64, Amer. Math. Soc., Providence, RI, 1987, pp. 123–132.

[57] O. Müller, *The Cauchy problem of Lorentzian minimal surfaces in globally hyperbolic manifolds*, Ann. Global Anal. Geom. **32** (2007), no. 1, 67–85.

[58] J. C. Neu, *Kinks and the minimal surface equation in Minkowski space*, Phys. D **43** (1990), no. 2-3, 421–434.

[59] L. Nguyen and G. Tian, *On smoothness of timelike maximal cylinders in three-dimensional vacuum spacetimes*, Classical Quantum Gravity **30** (2013), no. 16, 165010, 26.

[60] J. F. Nye, *Natural Focusing and Fine Structure of Light: Caustics and Wave Dislocations*, Institute of Physics Publishing, Bristol, 1999.

[61] R. Osserman, *Proof of a conjecture of Nirenberg*, Comm. Pure Appl. Math. **12** (1959), 229–232.

[62] ———, *Global properties of classical minimal surfaces*, Duke Math. J. **32** (1965), 565–573.

[63] ———, *A Survey of Minimal Surfaces*, second ed., Dover publications, inc. New York, 1986.

[64] A. Pais, *Subtle is the Lord: The Science and the Life of Albert Einstein*, Oxford University Press, 1982.

[65] E. A. Paxton, *Embeddedness of timelike maximal surfaces in (1+2)-Minkowski space*, (2019), preprint, `https://arxiv.org/abs/1902.08952`.

[66] G. P. Pron'ko, A. V. Razumov, and L. D. Solov'ev, *Classical dynamics of a relativistic string*, Sov. J. Part. Nucl. **14** (1983), no. 3, 229–237.

[67] D. E. Rowe, *A look back at Hermann Minkowski's Cologne lecture "Raum und Zeit"*, Math. Intelligencer **31** (2009), no. 2, 27–39.

[68] J. M. I. Shatah and M. Struwe, *Geometric Wave Equations*, Courant lecture notes in mathematics, Courant Institute of Mathematical Sciences, 2000.

[69] Q.-Y. Sun, *Global classical solutions of initial-boundary value problem for the equations of time-like extremal surfaces in the Minkowski space*, Pure Appl. Math. Q. **8** (2012), no. 3, 781–804.

[70] T. Weinstein, *An Introduction to Lorentz Surfaces*, Walter de Gruyter, 1996.

[71] E. W. Weisstein, *Enneper's minimal surface*, MathWorld–A Wolfram web resource.

[72] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410.

[73] W. W. Y. Wong, *Stability and instability of expanding solutions to the Lorentzian constant-positive-mean-curvature flow*, (2014), preprint, `https://arxiv.org/pdf/1404.0223v3.pdf`.

[74] ———, *Global existence for the minimal surface equation on* $\mathbb{R}^{1,1}$, Proc. Amer. Math. Soc. **B4** (2017), 47–52.

[75] ———, *Singularities of axially symmetric time-like minimal submanifolds in Minkowski space*, J. Hyperbolic Differ. Eq. **15** (2018), no. 1, 1–13.

[76] W. Yan, *Nonlinear stability of explicit self-similar solutions for the timelike extremal hypersurfaces in* $\mathbb{R}^{1+3}$, (2019), preprint, `https://arxiv.org/abs/1907.01126`.

[77] B. Zwiebach, *A First Course in String Theory*, Cambridge University Press, 2003.