Embeddedness of Timelike Maximal Surfaces in (1+2) Minkowski Space

arxiv.org/abs/1902.08952

(to appear in Annales Henri Poincaré)

slides available @ adampaxton9973.github.io/my-page

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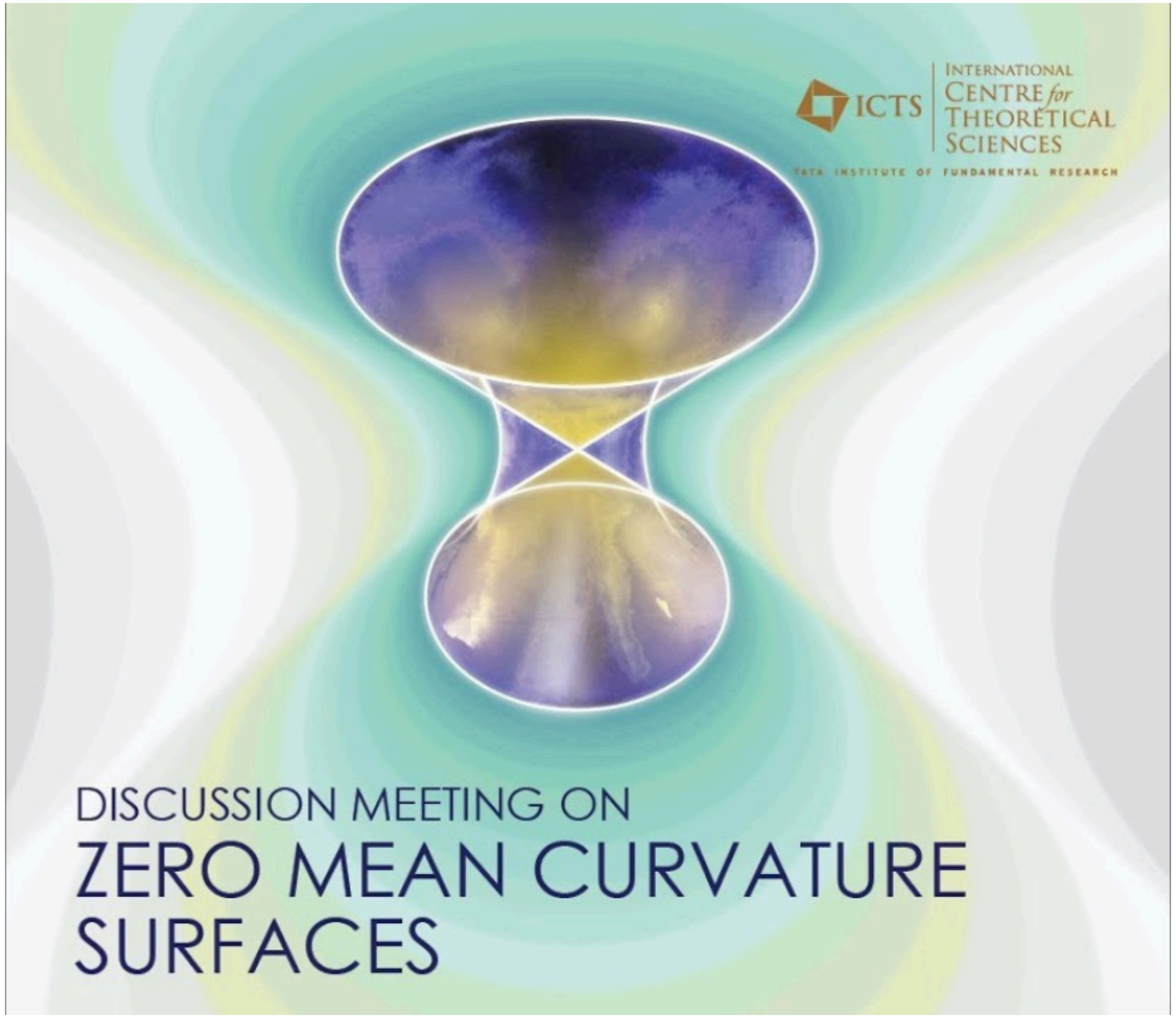

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Rough plan of talk:

1. A (Very) Brief Tour of Minimal Surfaces in $\mathbb{R}^3$.

2. Review of Differential Geometry in $\mathbb{R}^{1+2}$ & the Cauchy Problem for Timelike Maximal(/Minimal) Surfaces.

3. Some new results:

> **Theorem 1 [P. 2019]:** *Every smooth properly immersed timelike maximal surface in $\mathbb{R}^{1+2}$ is embedded, and is a smooth graph over bounded subsets.*

> **Theorem 2 [P. 2019]:** *Singularity formation for a TMS always involves a curvature blow-up, with blow-up in an L^1L^∞ sense.*

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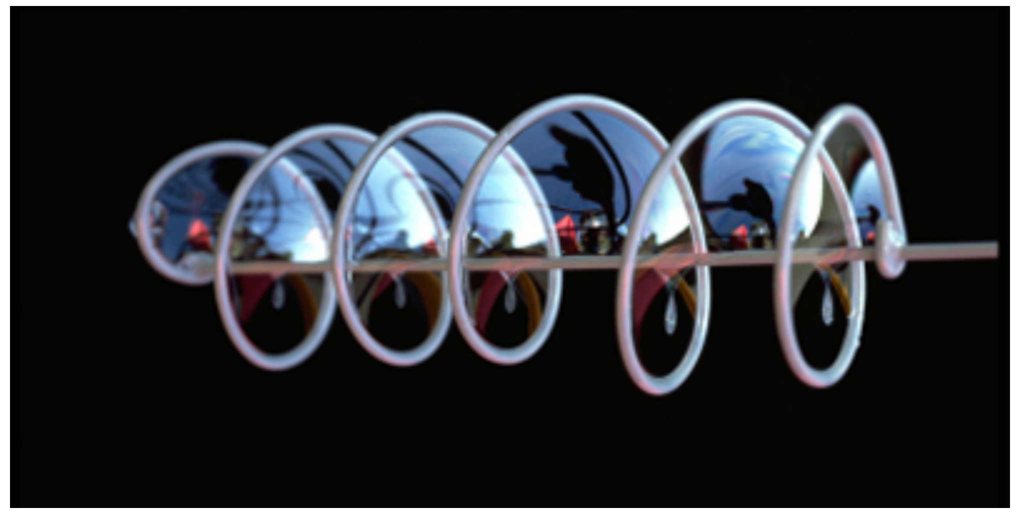
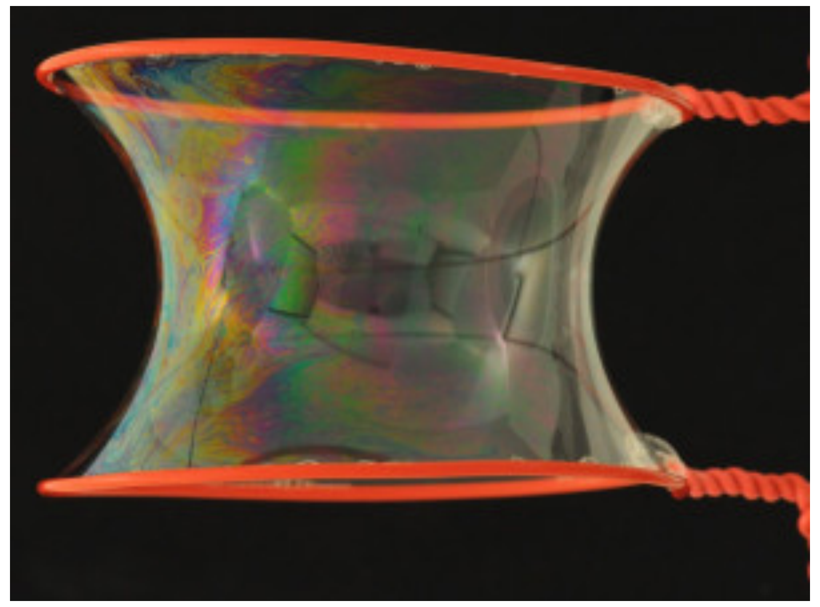
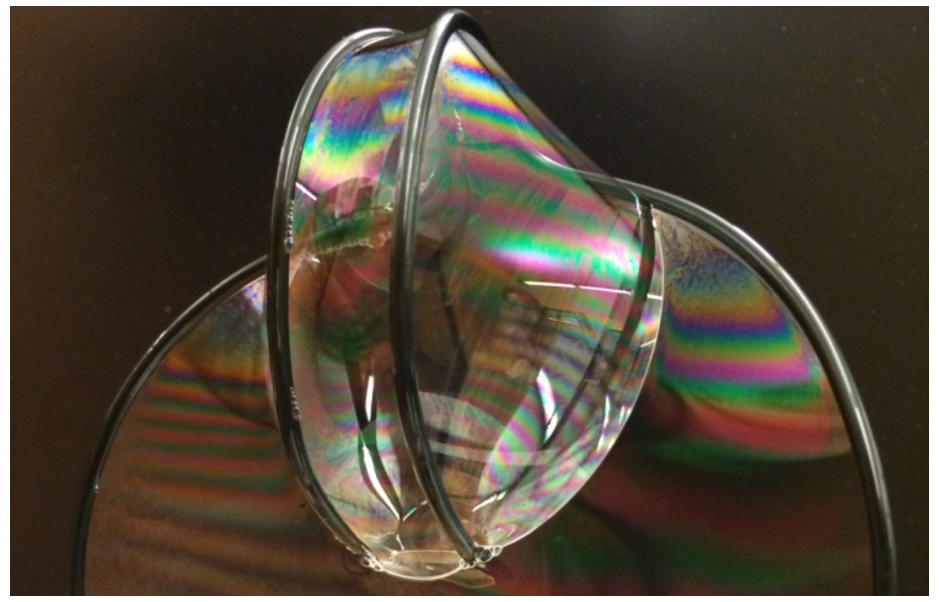
Part I: Minimal surfaces in $\mathbb{R}^3$

- In 1760 Lagrange wrote down the equation

$$(1.1)\quad \frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{\sqrt{1+\frac{\partial u}{\partial x}^2+\frac{\partial u}{\partial y}^2}}\right)+\frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\sqrt{1+\frac{\partial u}{\partial x}^2+\frac{\partial u}{\partial y}^2}}\right)=0$$

  describing a surface $\Sigma = \{(x, y, u(x, y))\} \subseteq \mathbb{R}^3$ which extremises the area functional (a minimal surface).

- Minimal surfaces describe soap films :)
- Lagrange didn't write down solutions to (1.1) apart from the plane ($D^2u \equiv 0$)

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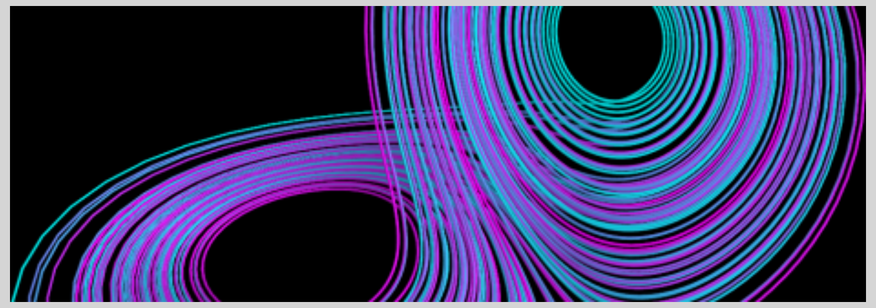
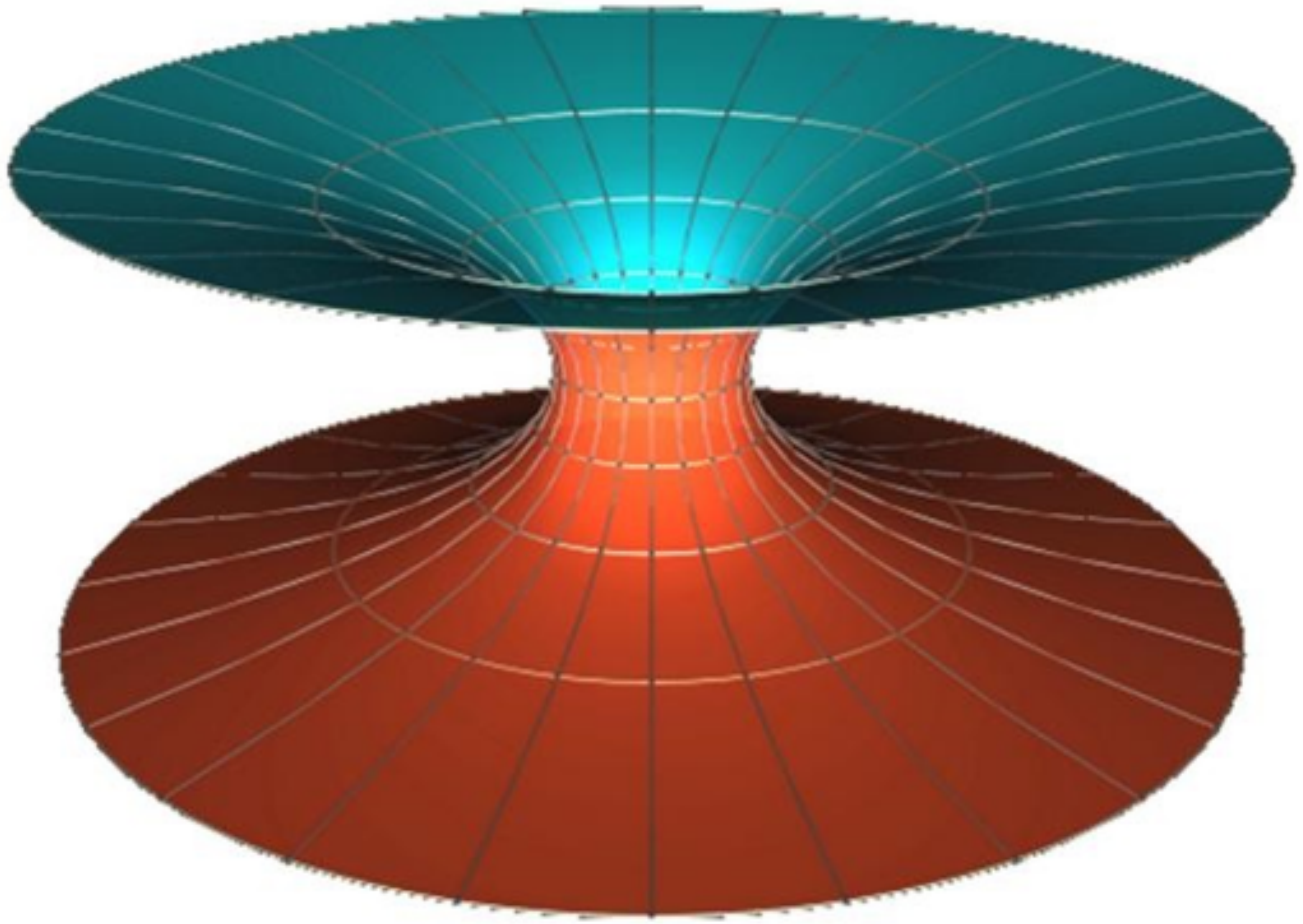
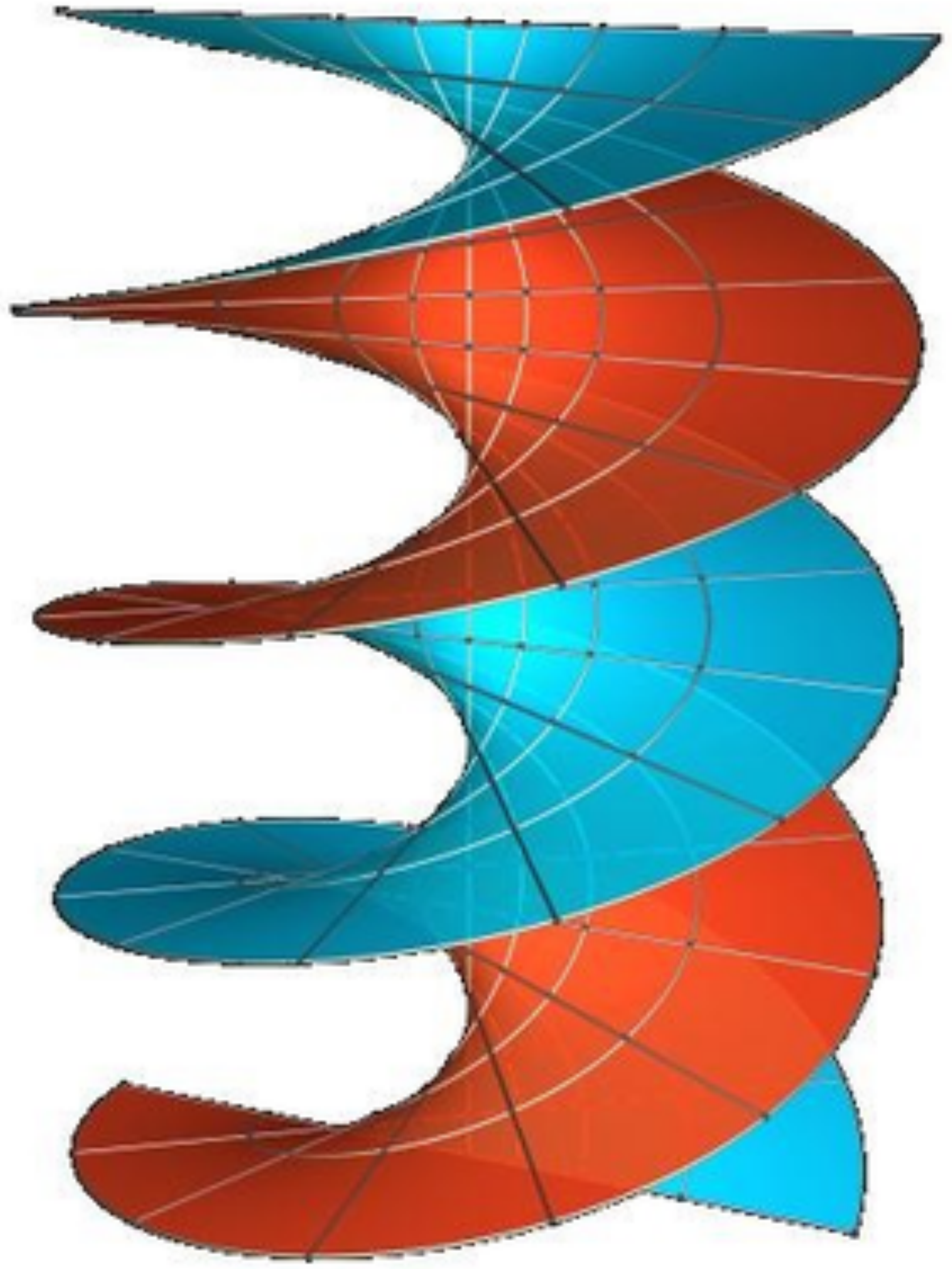


In 1776 Meusnier identified minimal surfaces as *surfaces of vanishing mean curvature,* and was able to give the first non-trivial examples:



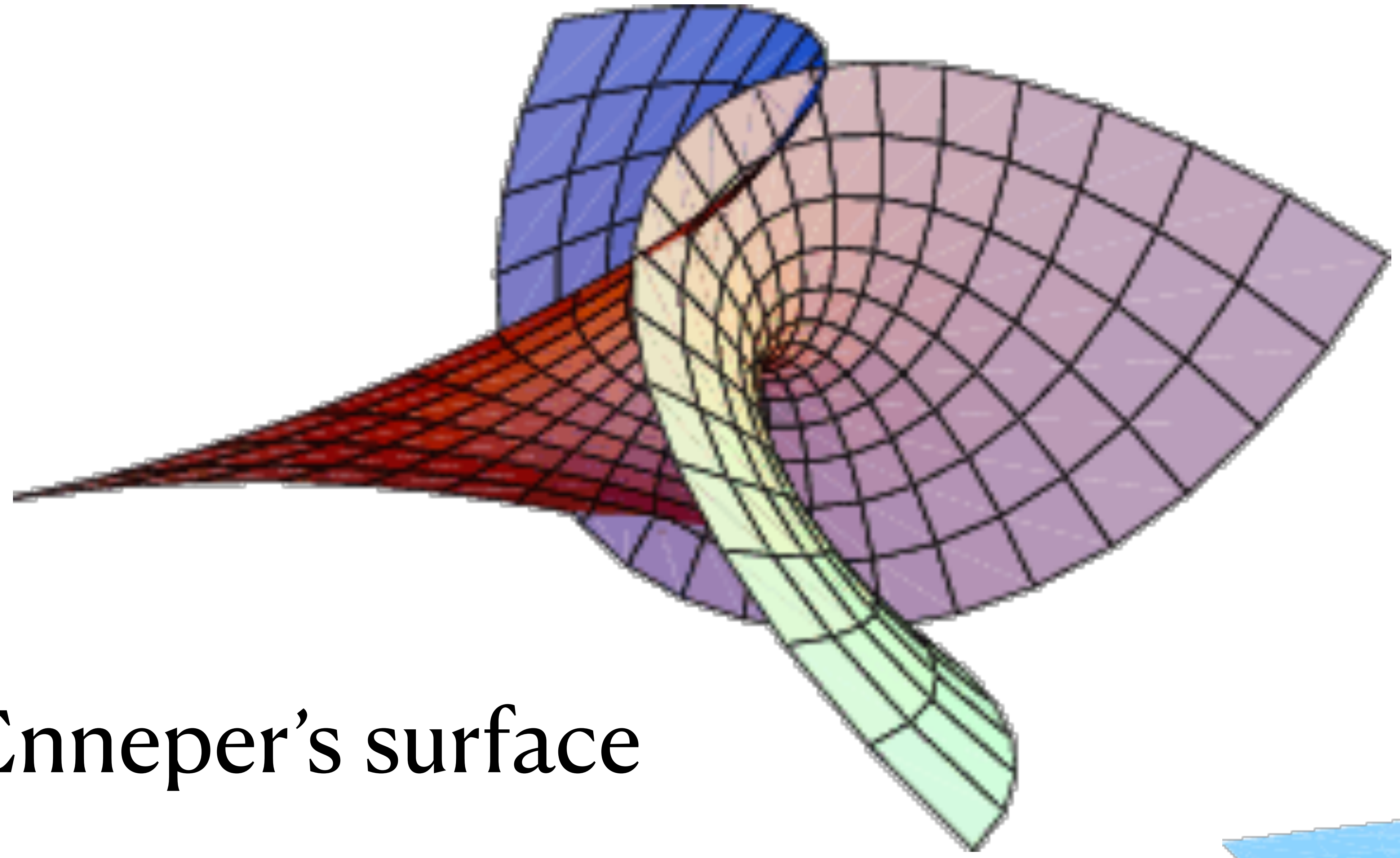
1. The catenoid



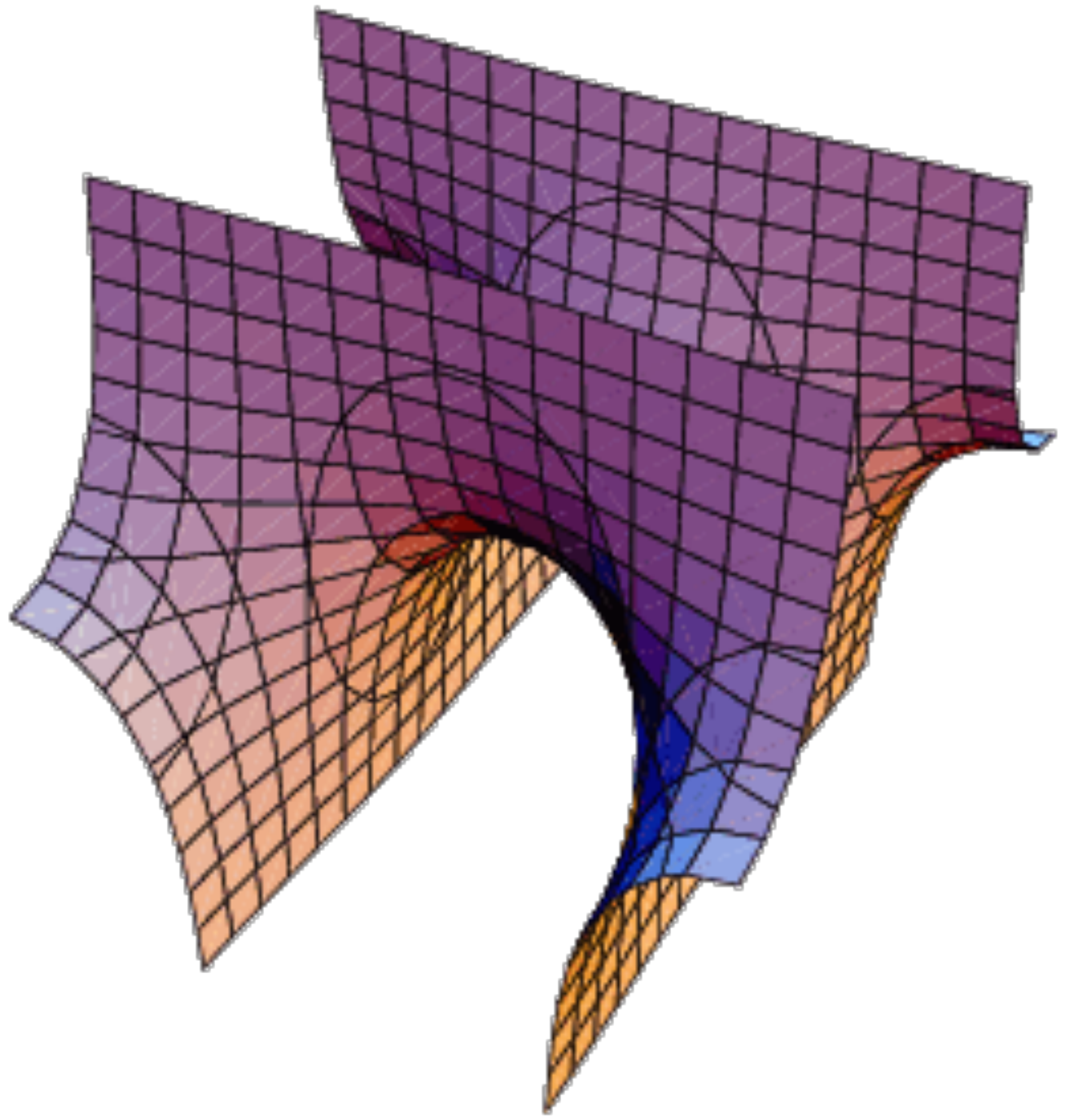
2. The helicoid

Images courtesy of M. Weber.

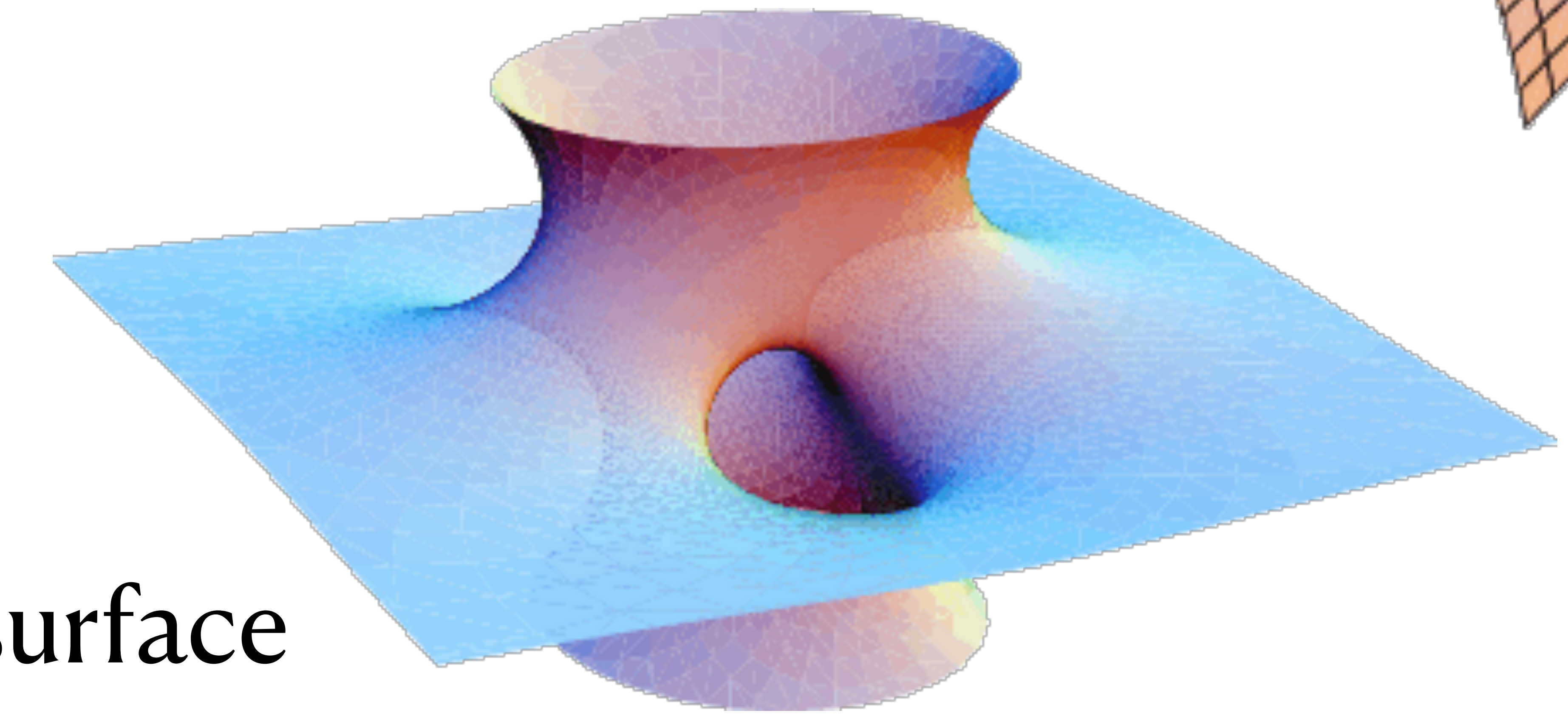
Over the last 250 years many more exciting examples of minimal surfaces in $\mathbb{R}^3$ have been found...



1. Enneper's surface



2. Scherk's surface



3. Costa's surface

Images courtesy of E. W. Weisstein

@ MathWorld–A Wolfram web resource

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And many beautiful theorems have been proved! (“Rigidity” of the plane)

Theorem (Bernstein 1915):

Any smooth complete minimal surface Σ in $\mathbb{R}^3$ which is a graph (i.e. $\Sigma = \{(x, y, u(x, y))\}$) must be a plane.

(I.e. there are no non-trivial solutions $u\colon \mathbb{R}^2 \to \mathbb{R}$ to (2.1))

Theorem (Fujimoto 1988, building upon Osserman 1959 & others):

If Σ is a complete minimal surface in $\mathbb{R}^3$ and $N\colon \Sigma \to S^2$ is its unit normal vector, then either:

1. Image(N) is a single point (i.e. Σ is a plane), or
2. Image(N) omits at most 4 points in S^2.

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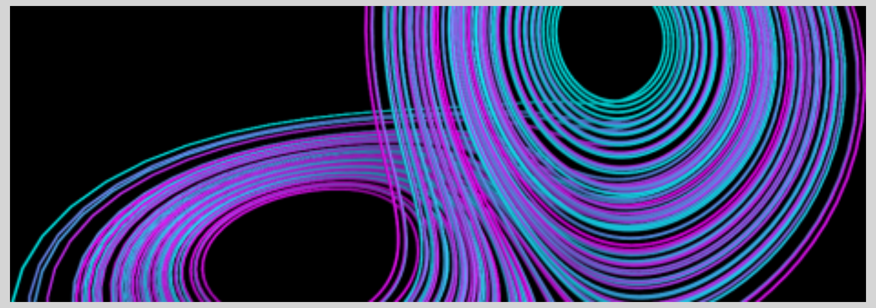

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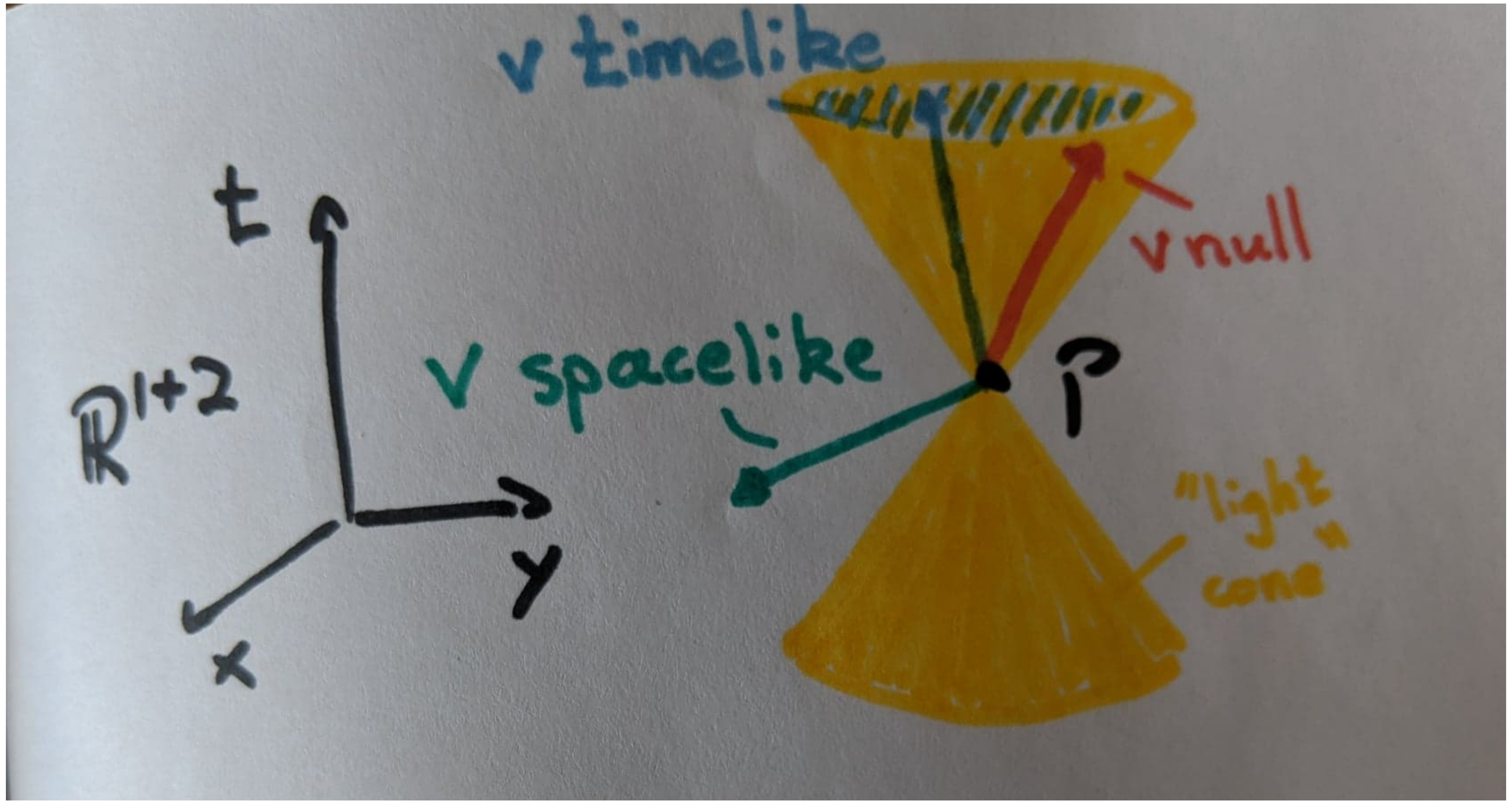
Part II: Minimal surfaces in $\mathbb{R}^{1+2}$

For vectors v & w in Minkowski space $\mathbb{R}^{1+2}$, the Minkowskian "inner product" is

$$m(v, w) := -v^0w^0 + v^1w^1 + v^2w^2$$

and we say

1. v is *timelike* if $m(v, v) < 0$
2. v is *spacelike* if $m(v, v) > 0$
3. v is *null* if $m(v, v) = 0$.

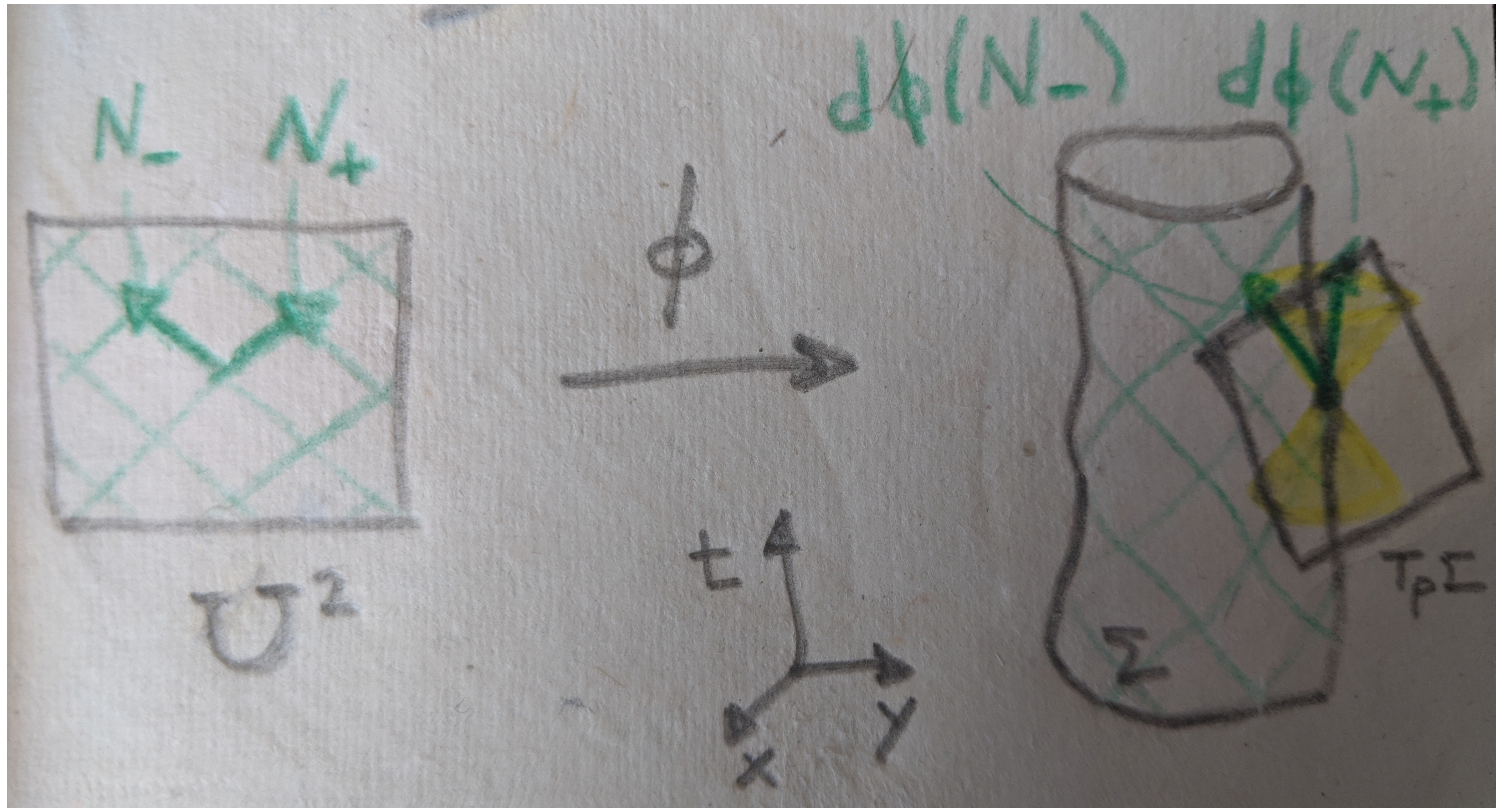


Suppose that U^2 is a surface and $\phi\colon\ U^2 \to \mathbb{R}^{1+2}$ is a smooth immersion. We define

$$g_{ij}(p) = m(\partial_i\phi, \partial_j\phi)$$

and say ϕ is *timelike* if $\det(g(p)) < 0$ for all $p \in U^2$ (i.e. g is Lorentzian).

$\iff$ there exist a pair of nowhere-zero, linearly independent vector fields $N_\pm$ on U^2 such that $d\phi(N_\pm)$ are null.



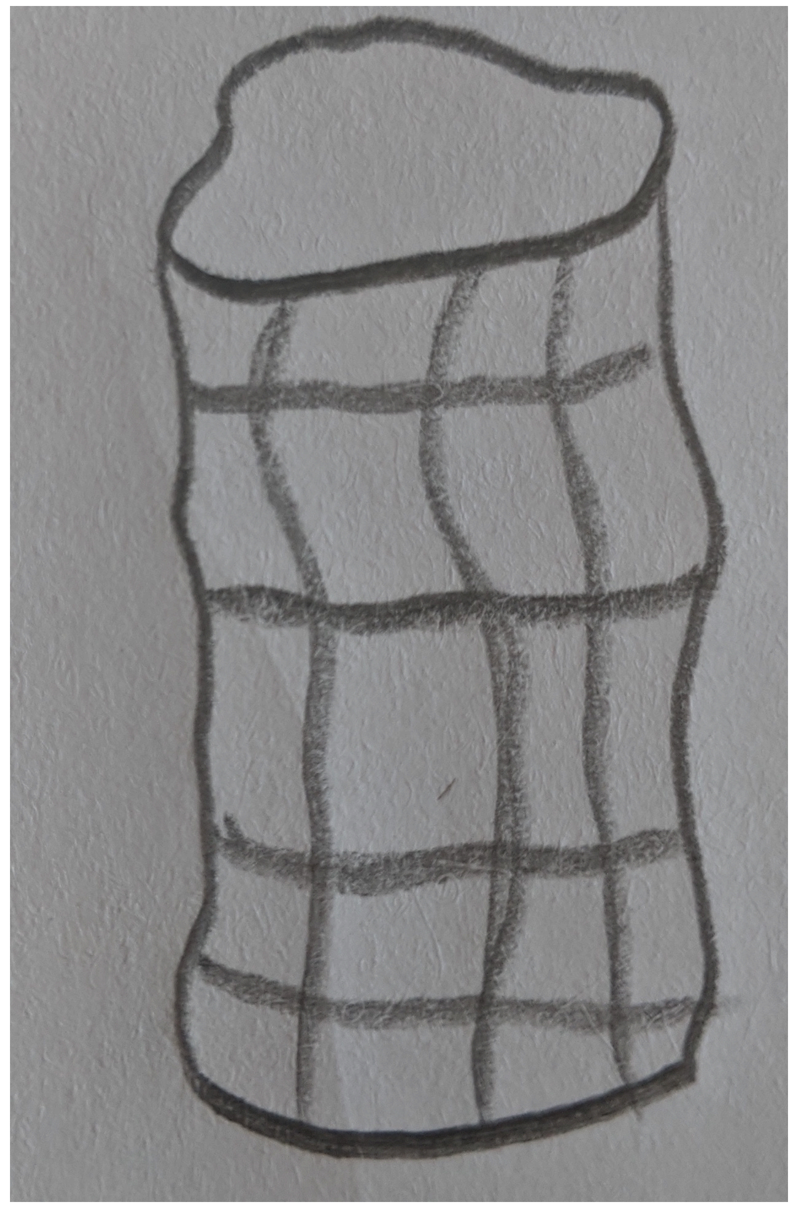
Morse Lemma (timelike surface=flow of curves): Suppose U^2 is connected and $\phi: U^2 \to \mathbb{R}^{1+2}$ is a smooth timelike immersion which is proper (i.e. the preimage of a compact set is compact). Then there exists a smooth diffeomorphism of the form

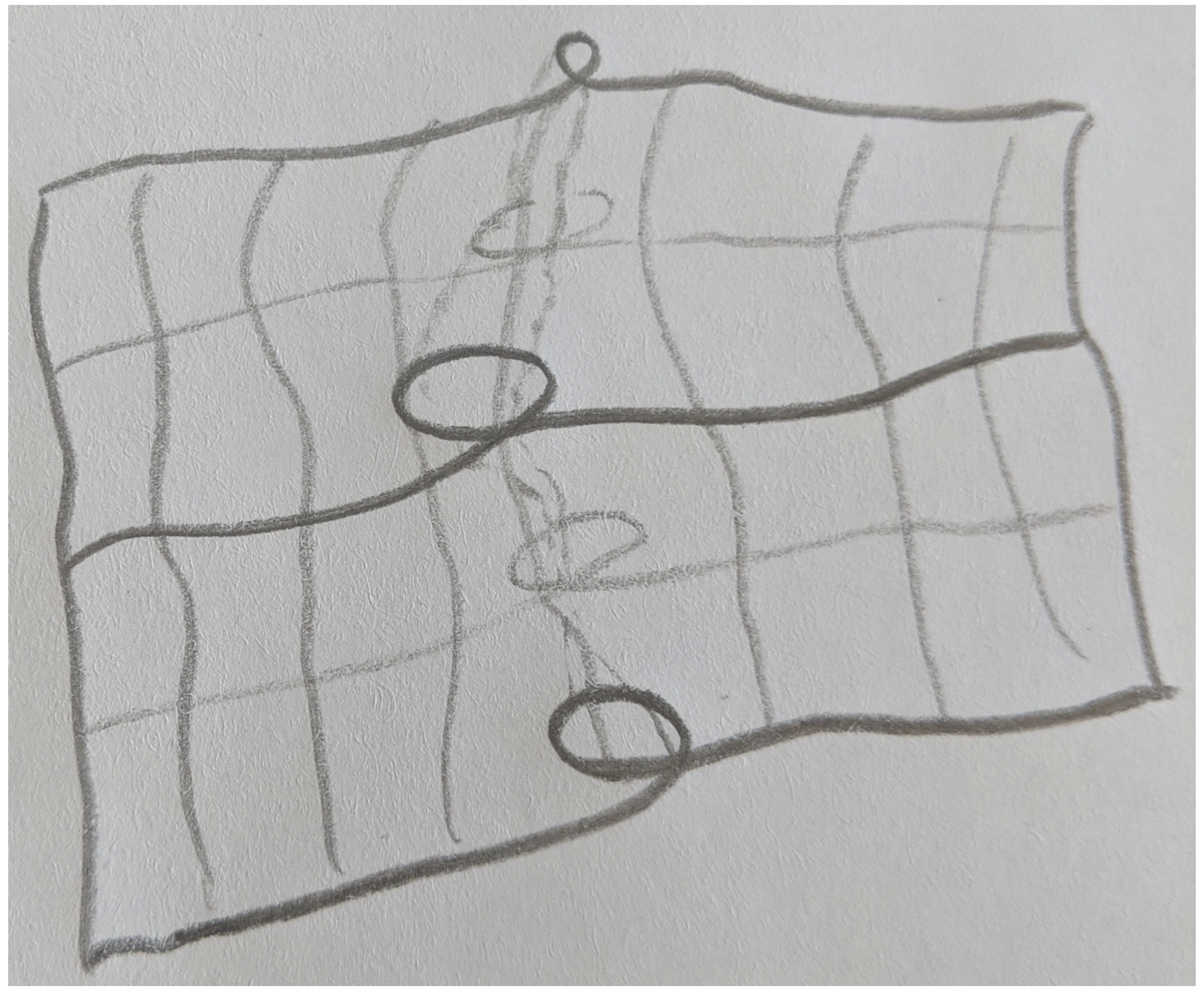
(i) $\psi: S^1 \times \mathbb{R} \to U^2$ or (ii) $\psi: \mathbb{R}^2 \to U^2$

such that, after diffeomorphism ($\phi \mapsto \phi \circ \psi$), one has $\phi(s,t) = (t, \gamma(s,t)) \in \mathbb{R}^{1+2}$.

In other words, properly immersed timelike surfaces come in 2 flavours:

1. cylinders (spatially compact), or
2. planes (spatially non-compact)





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A smooth timelike immersion $\phi\colon\ U^2 \to \mathbb{R}^{1+2}$ is maximal if

$$H(\phi) = \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|}g^{ij}\partial_j\phi\right) = 0$$

For a surface parameterised as a graph $\phi(x,t) = (t, x, u(x,t))$ this reads

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{\sqrt{1+{\frac{\partial u}{\partial x}}^2 - {\frac{\partial u}{\partial t}}^2}}\right) - \frac{\partial}{\partial t}\left(\frac{\frac{\partial u}{\partial t}}{\sqrt{1+{\frac{\partial u}{\partial x}}^2 - {\frac{\partial u}{\partial t}}^2}}\right) = 0$$

[Born-Infeld equation]

And for a surface parameterised in *isothermal* coordinates (s,t) (i.e. $g(s,t) = \rho(s,t)(-dt^2 + ds^2)$) it reads:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial s^2} = 0$$

In any case, we have a wave equation! The natural problem is thus the Cauchy problem.

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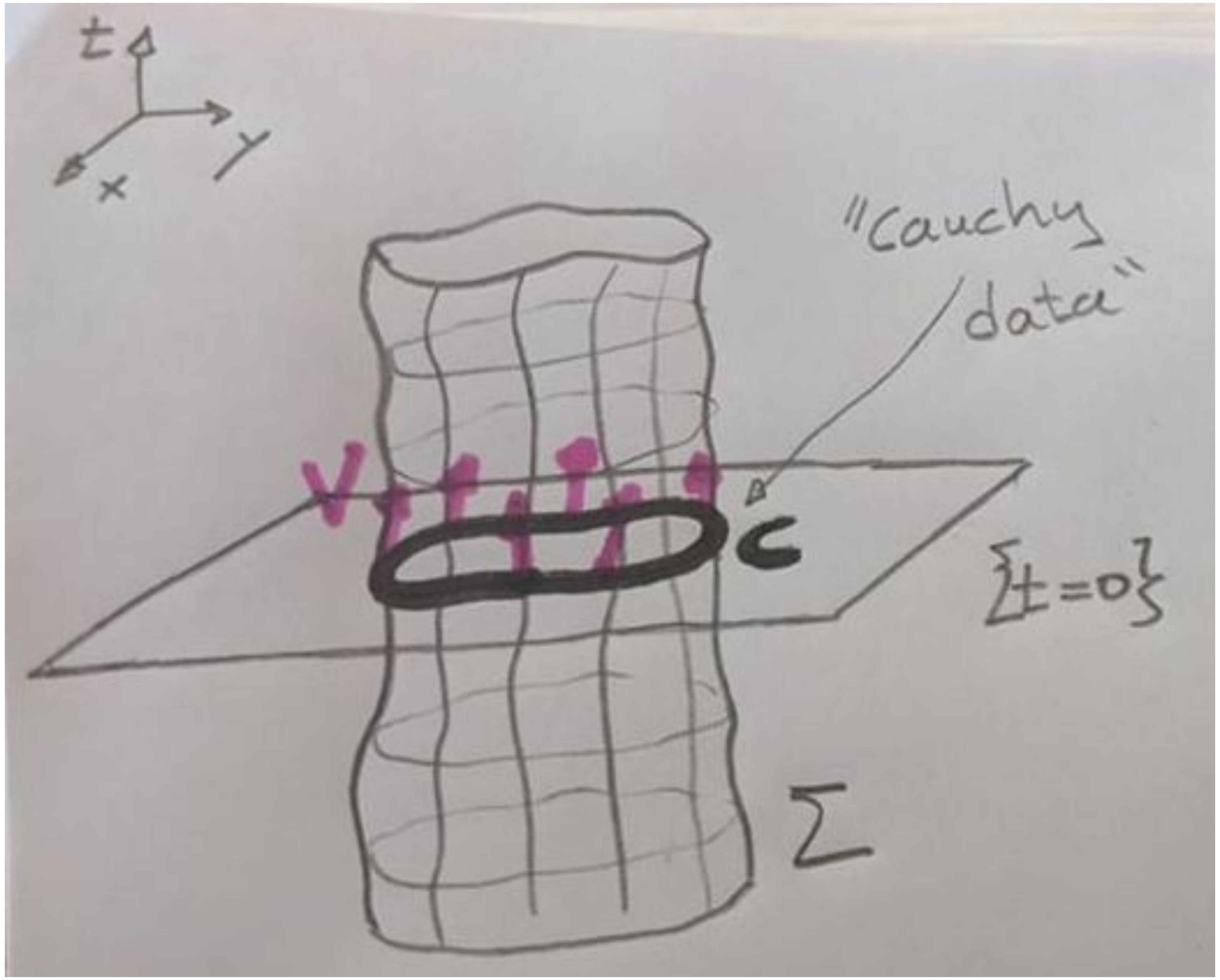
Cauchy problem:

Given a smooth immersed curve C in the plane $\{t = 0\}$ and a smooth timelike vector field V along C, find a smooth immersed timelike maximal surface Σ which intersects C and is tangent to V along C.

- A *global* solution is a proper immersion.

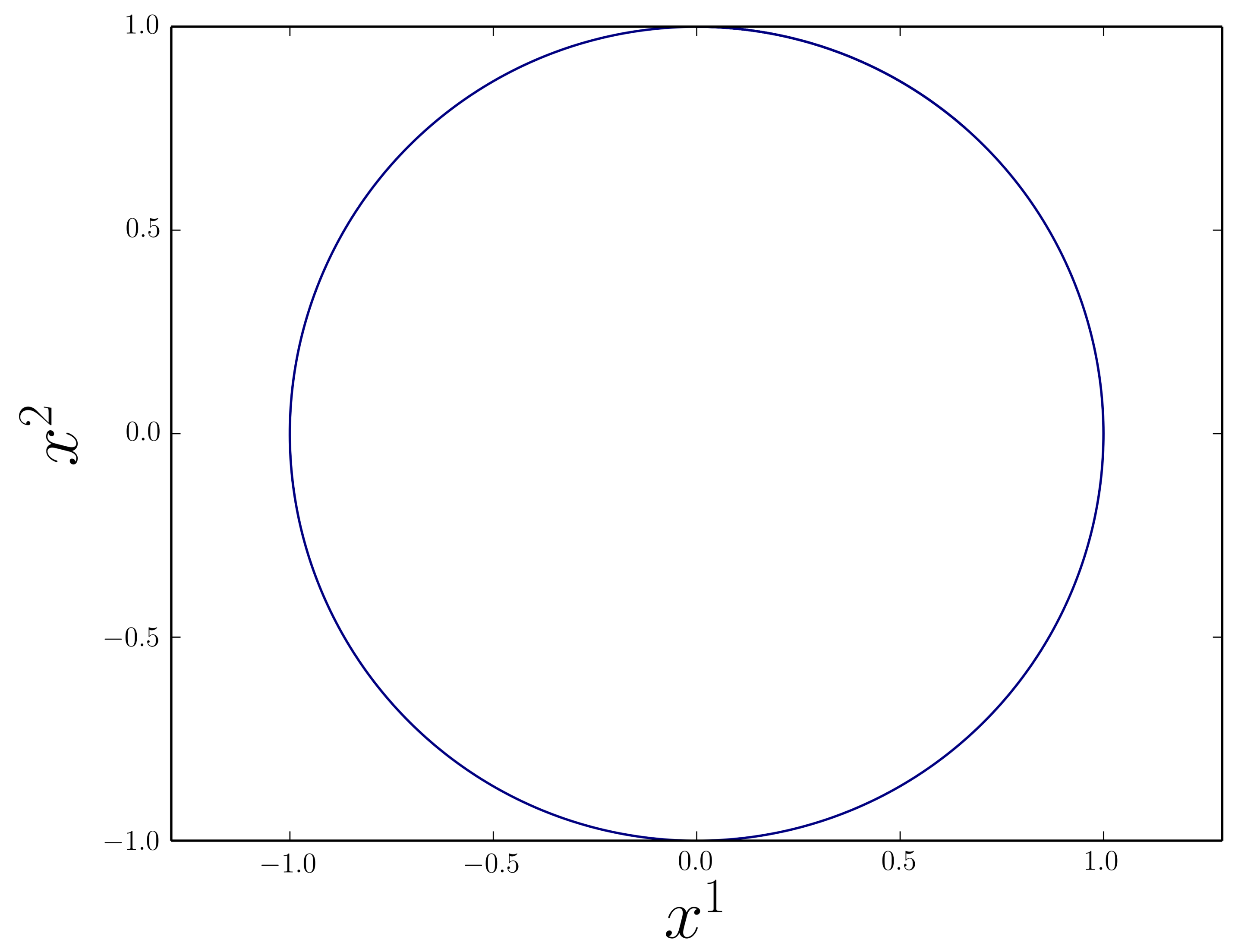


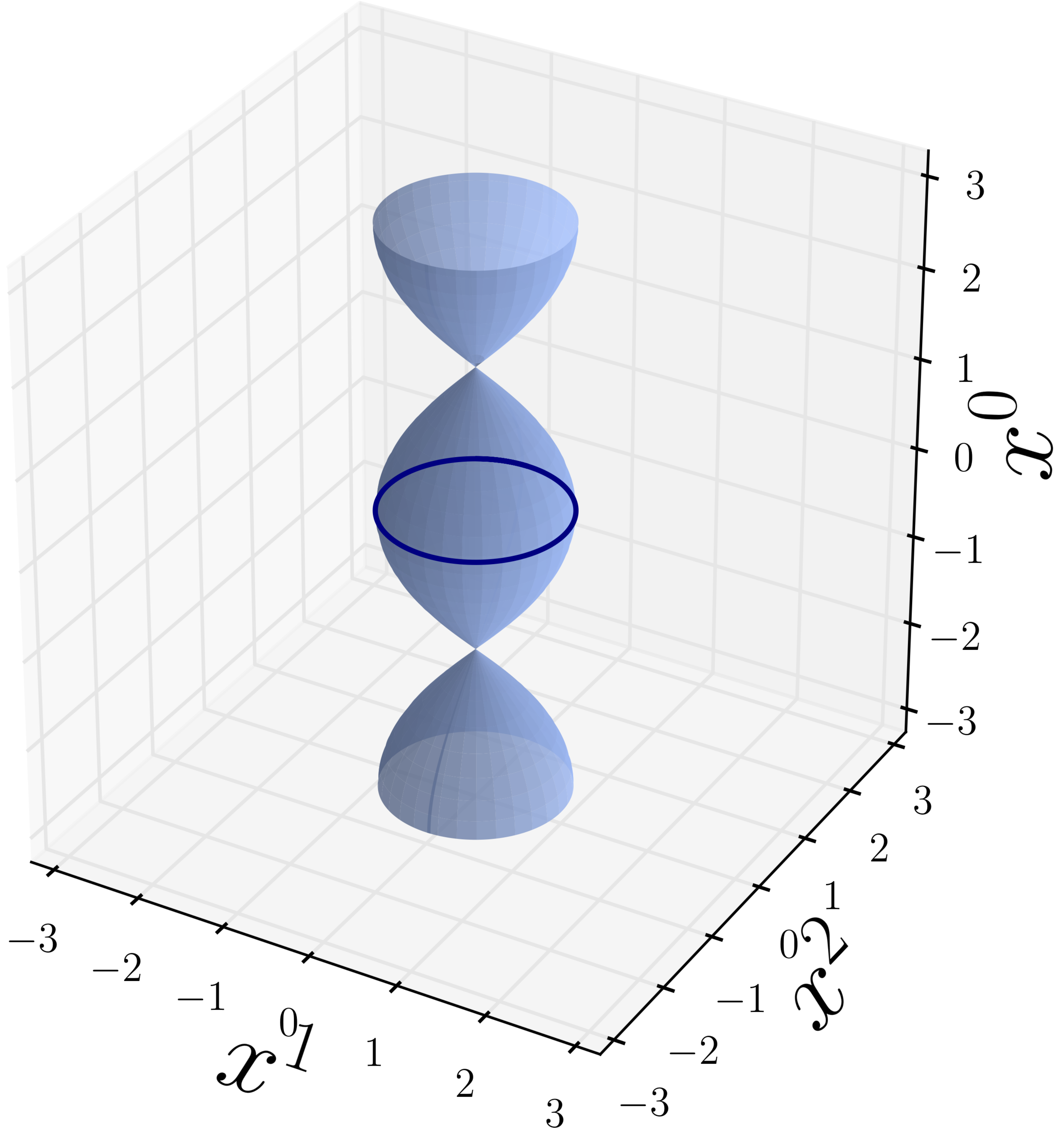
Example (The Shrinking Circle):

Let $C(s) = (0, \cos s, \sin s)$ and $V(s) = (1,0,0)$.

A Cauchy evolution of (C, V) is

$$\phi(s,t) = (t, \cos t \cos s, \cos t \sin s).$$





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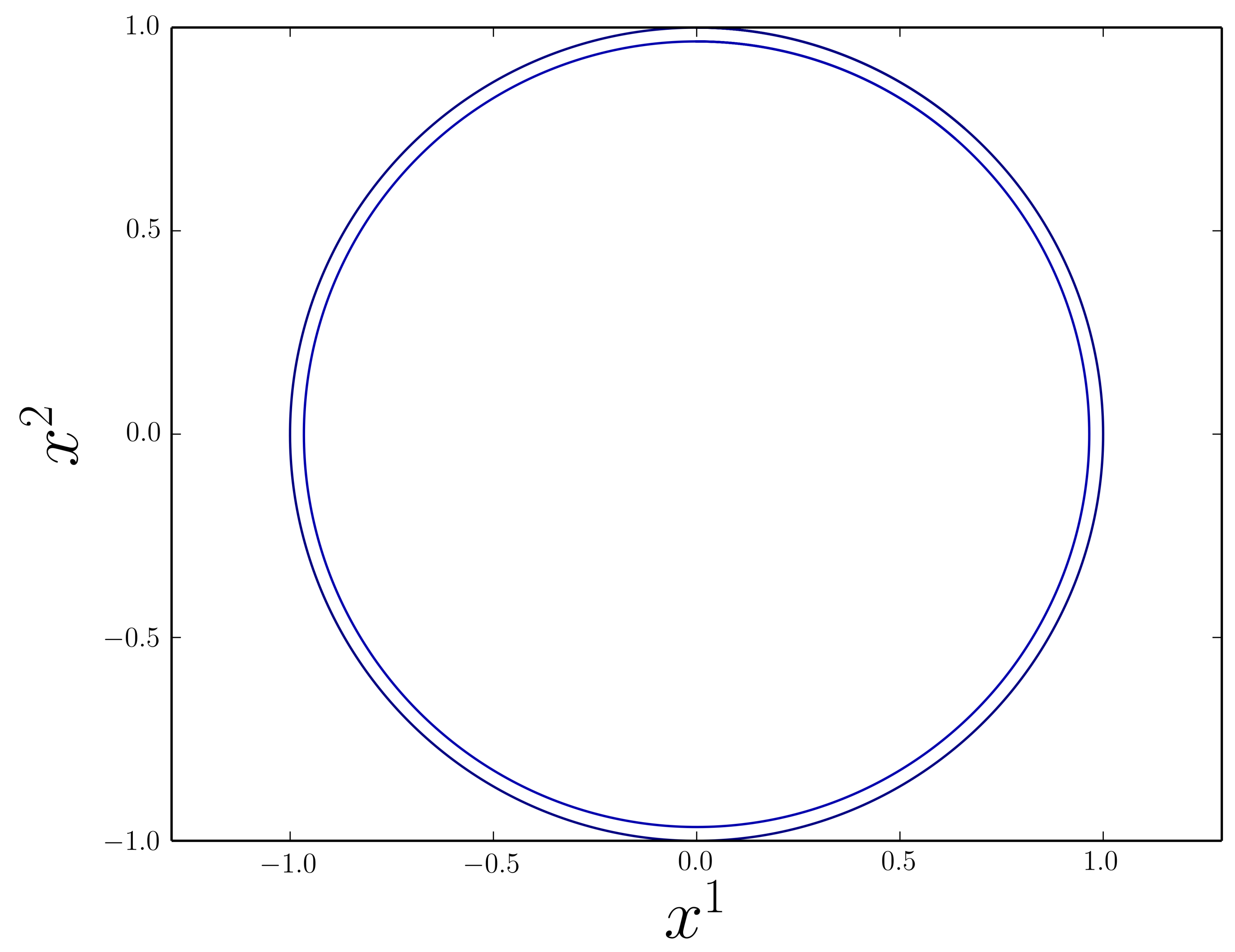

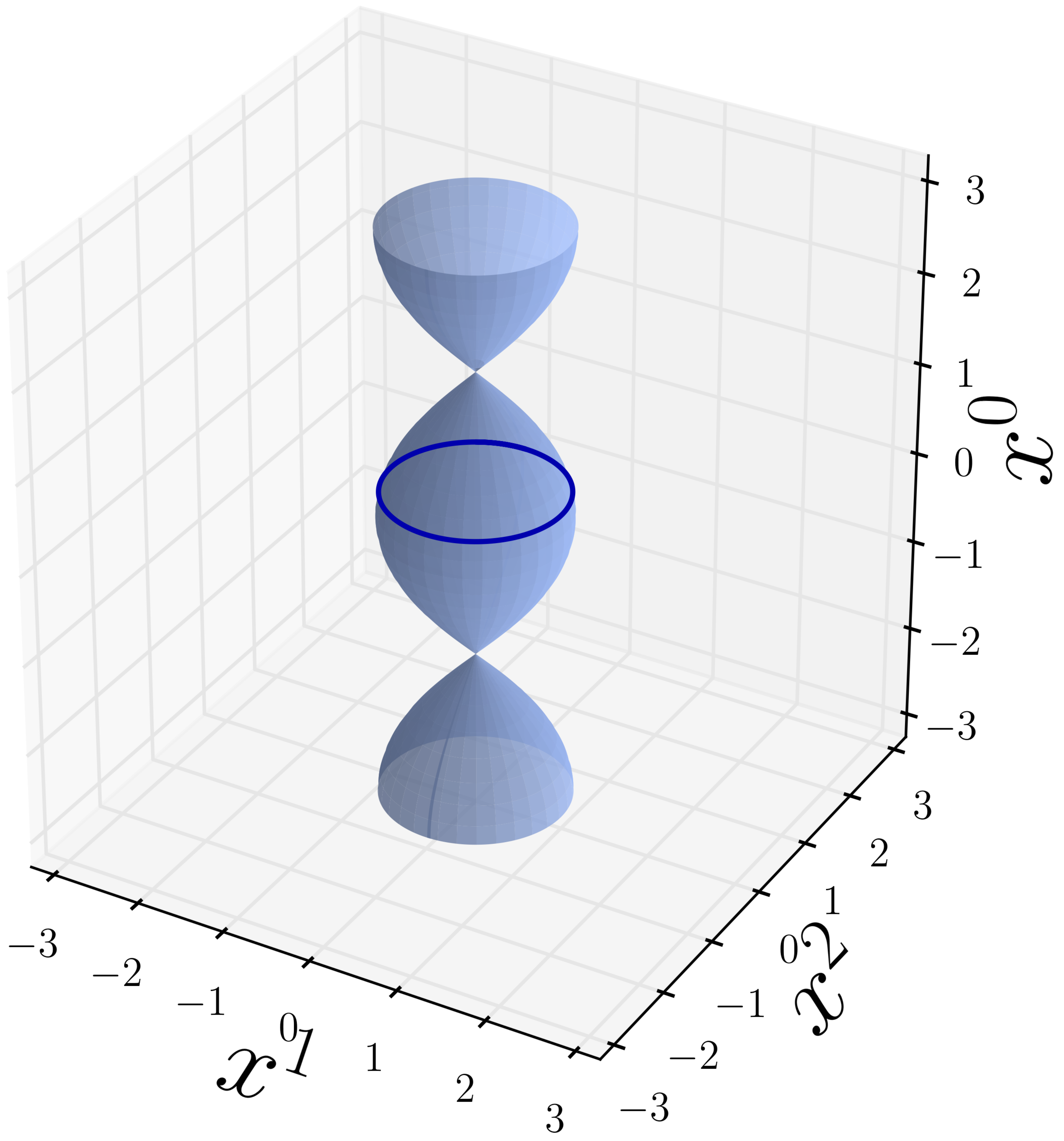

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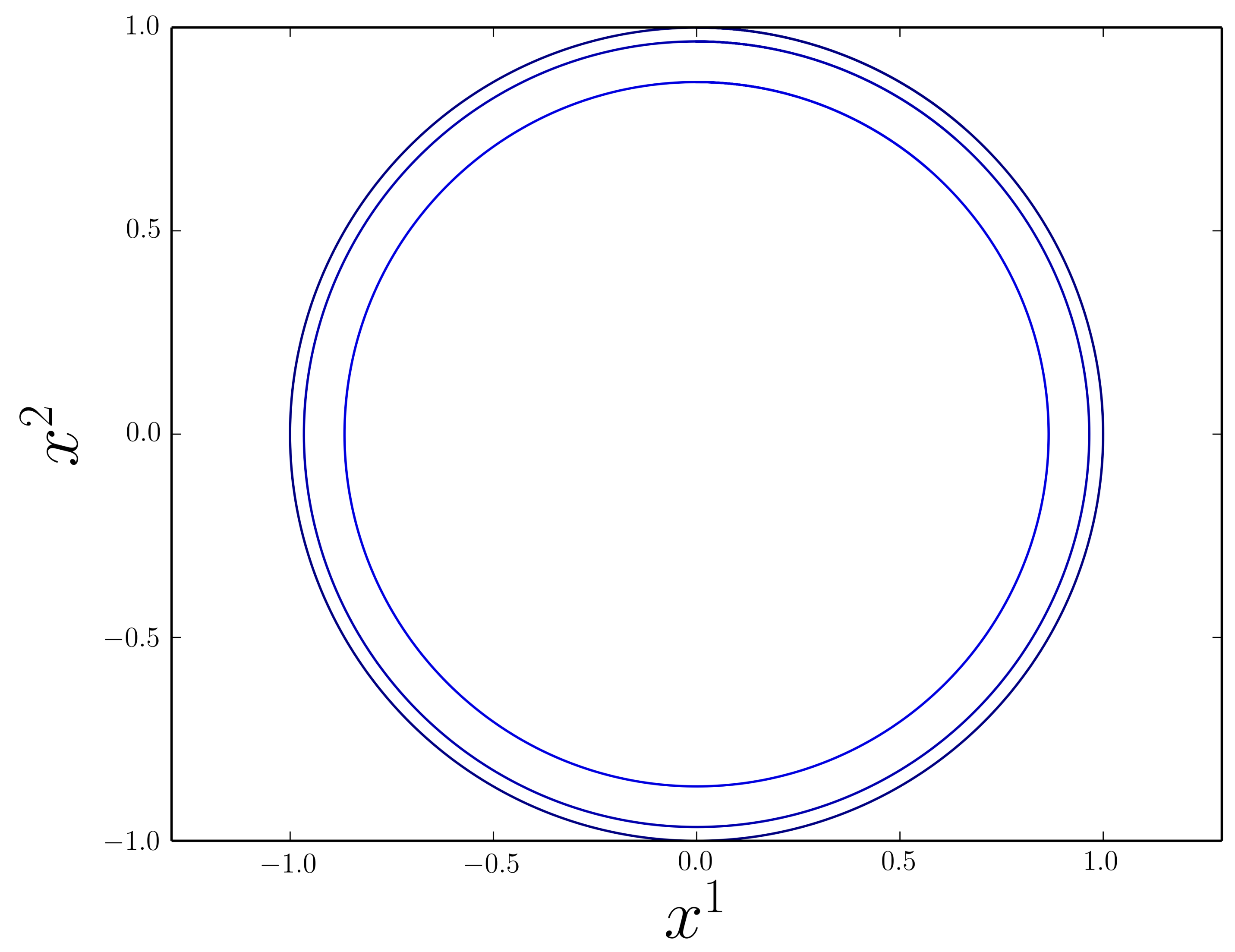

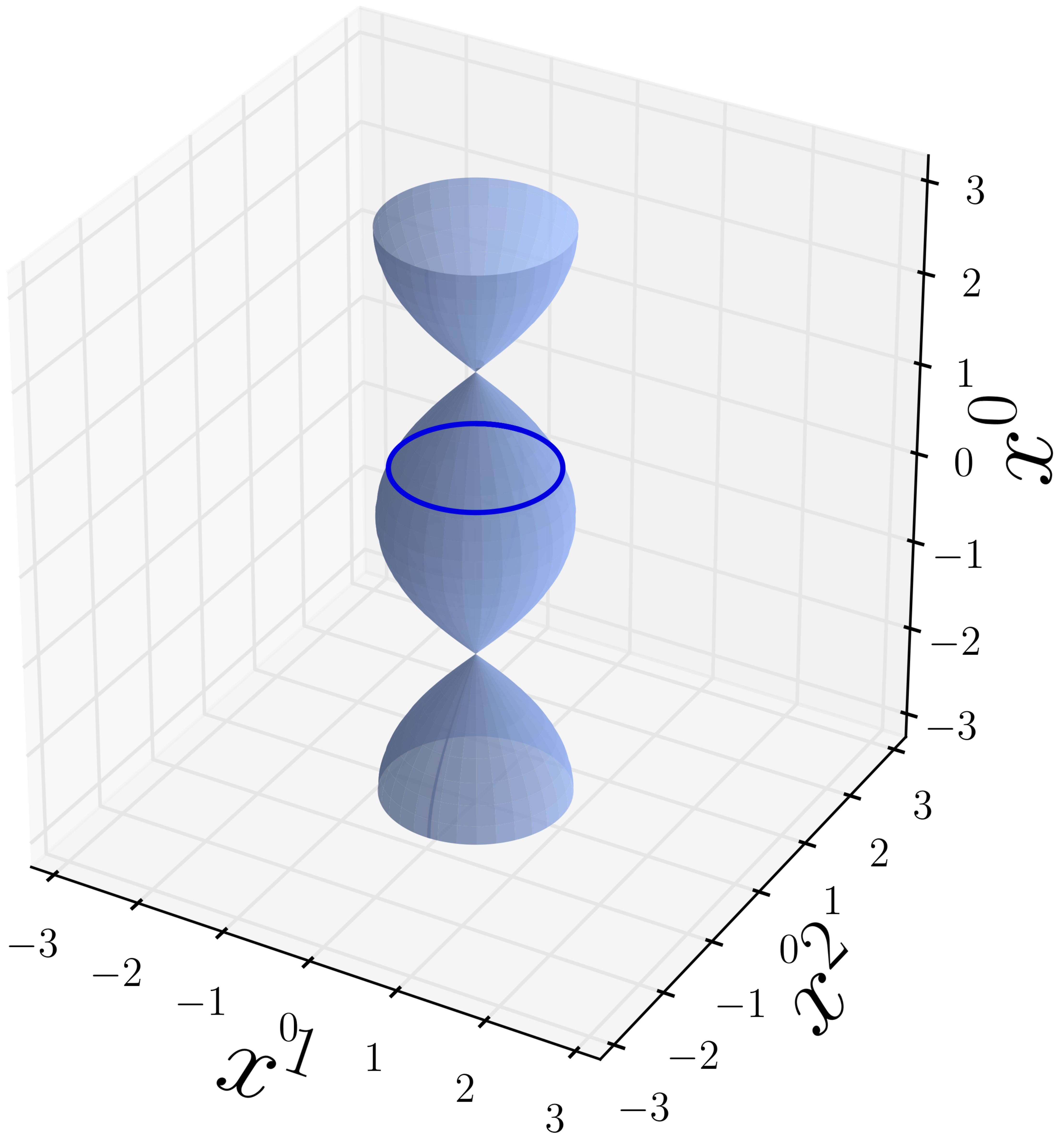

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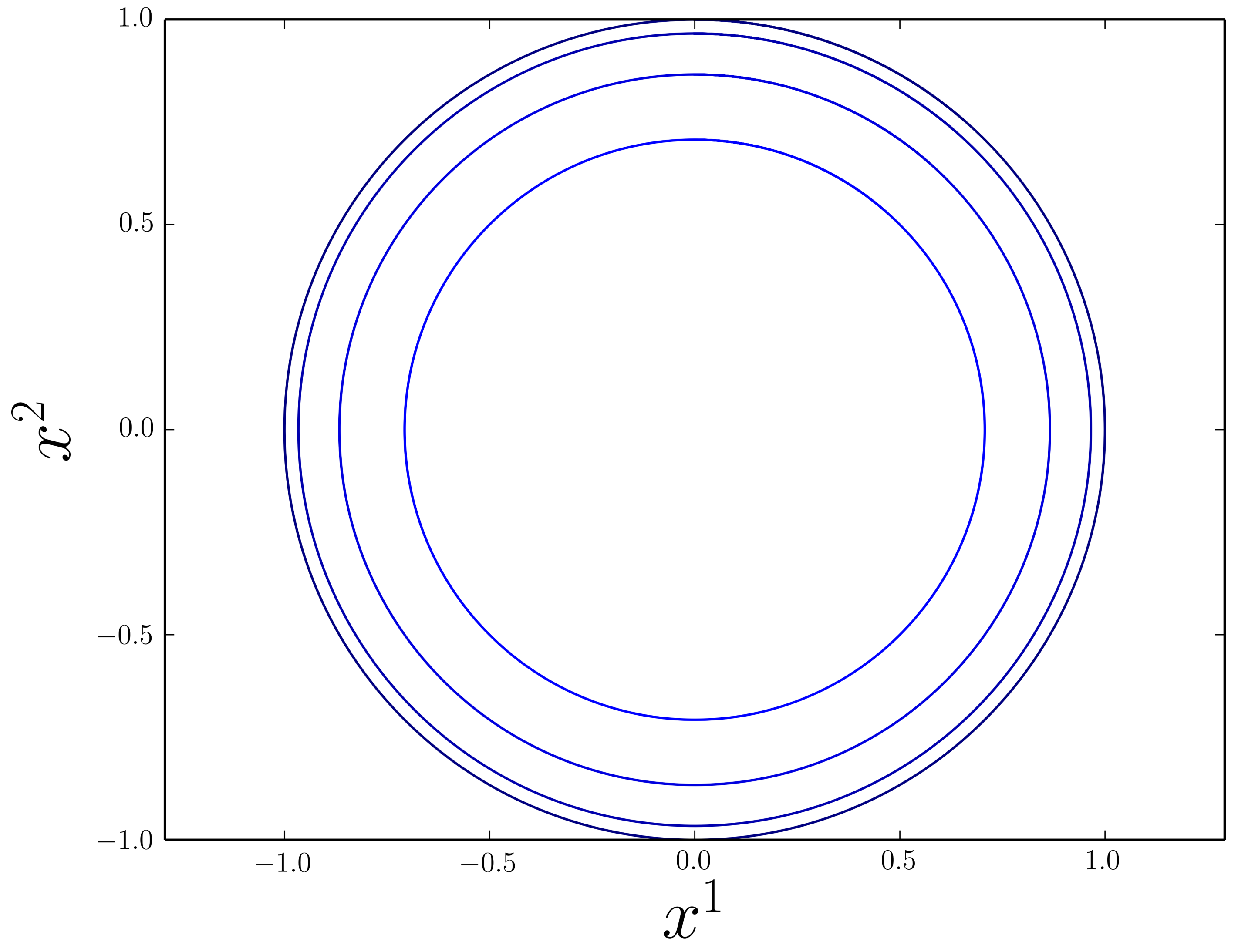

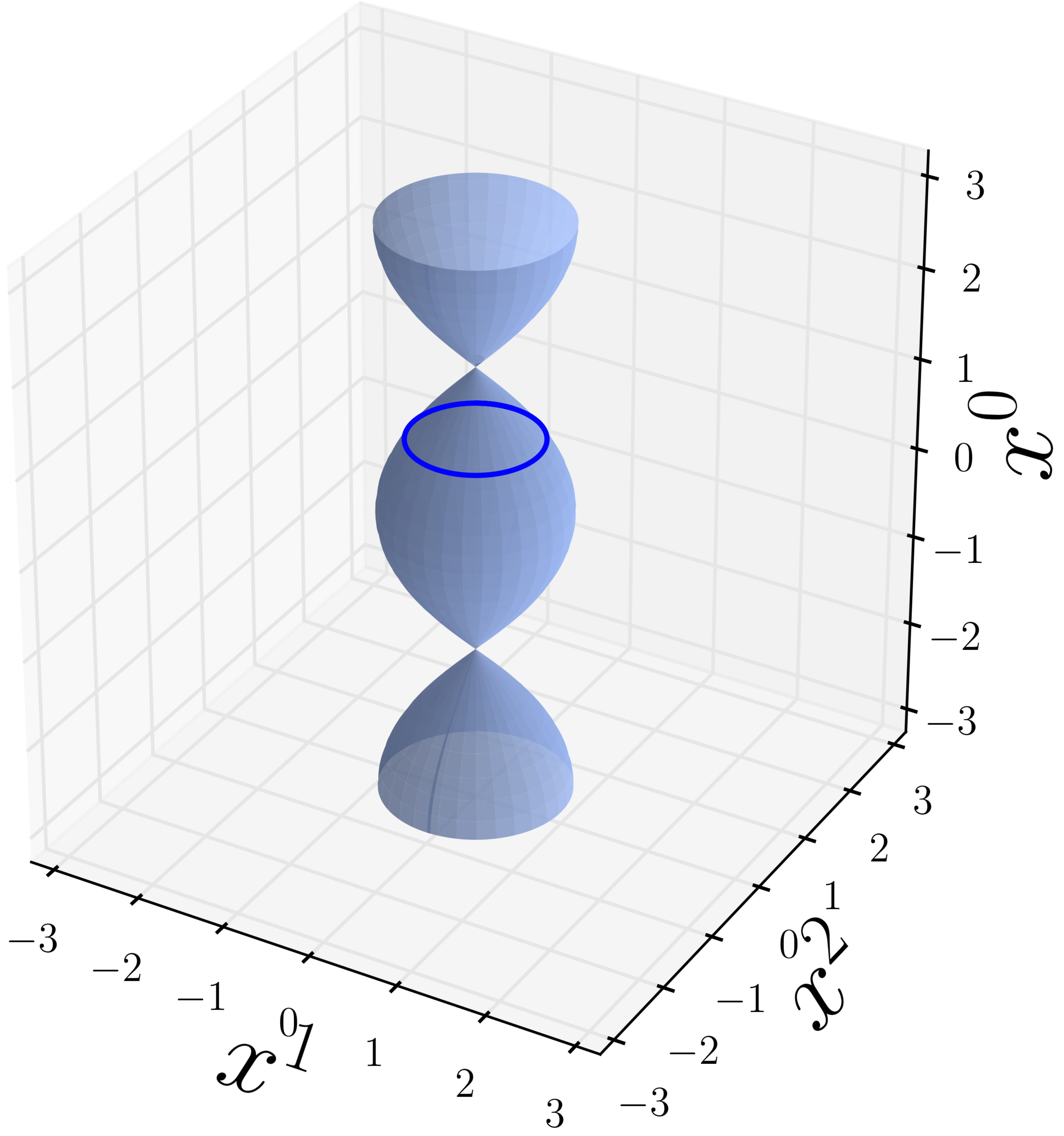

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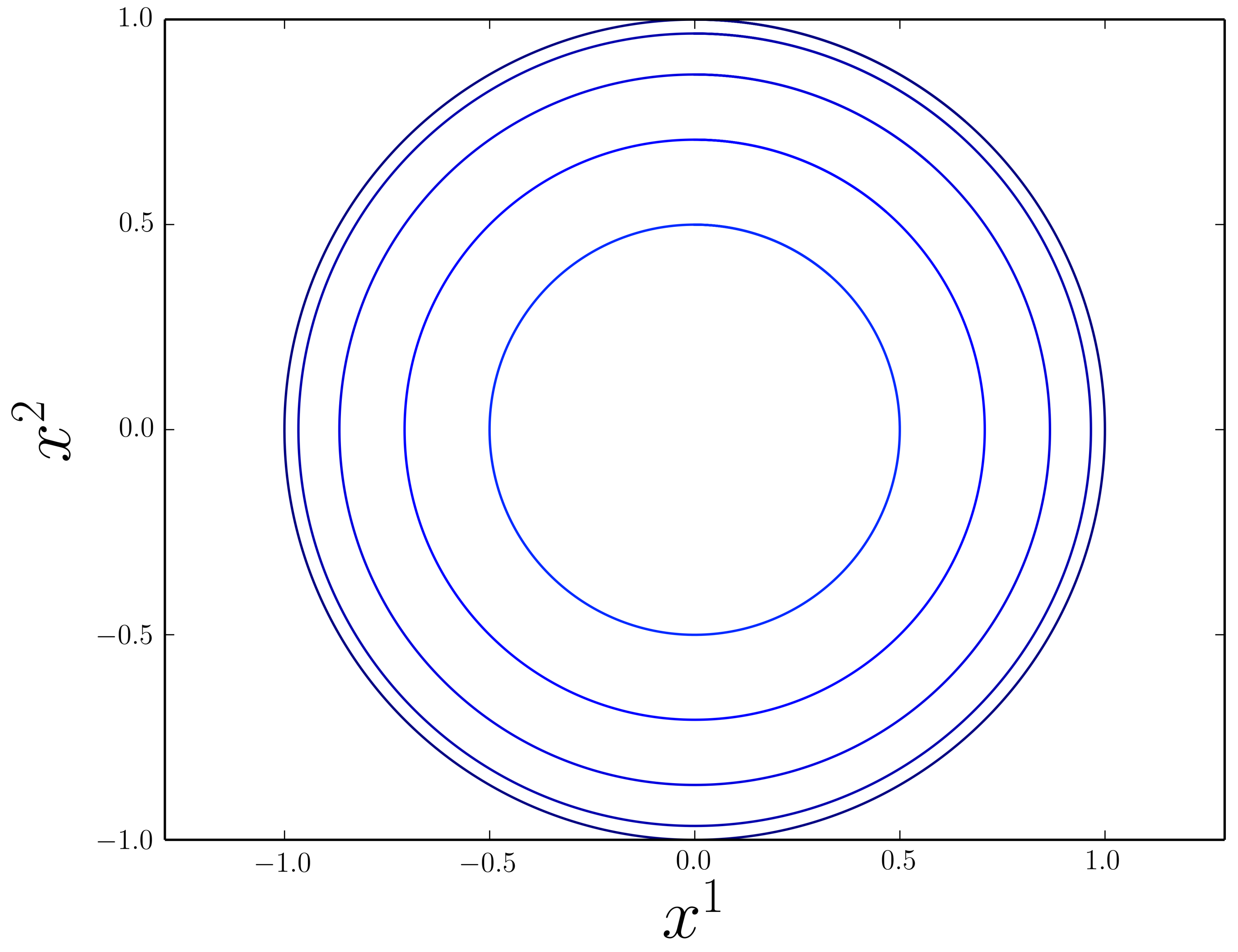

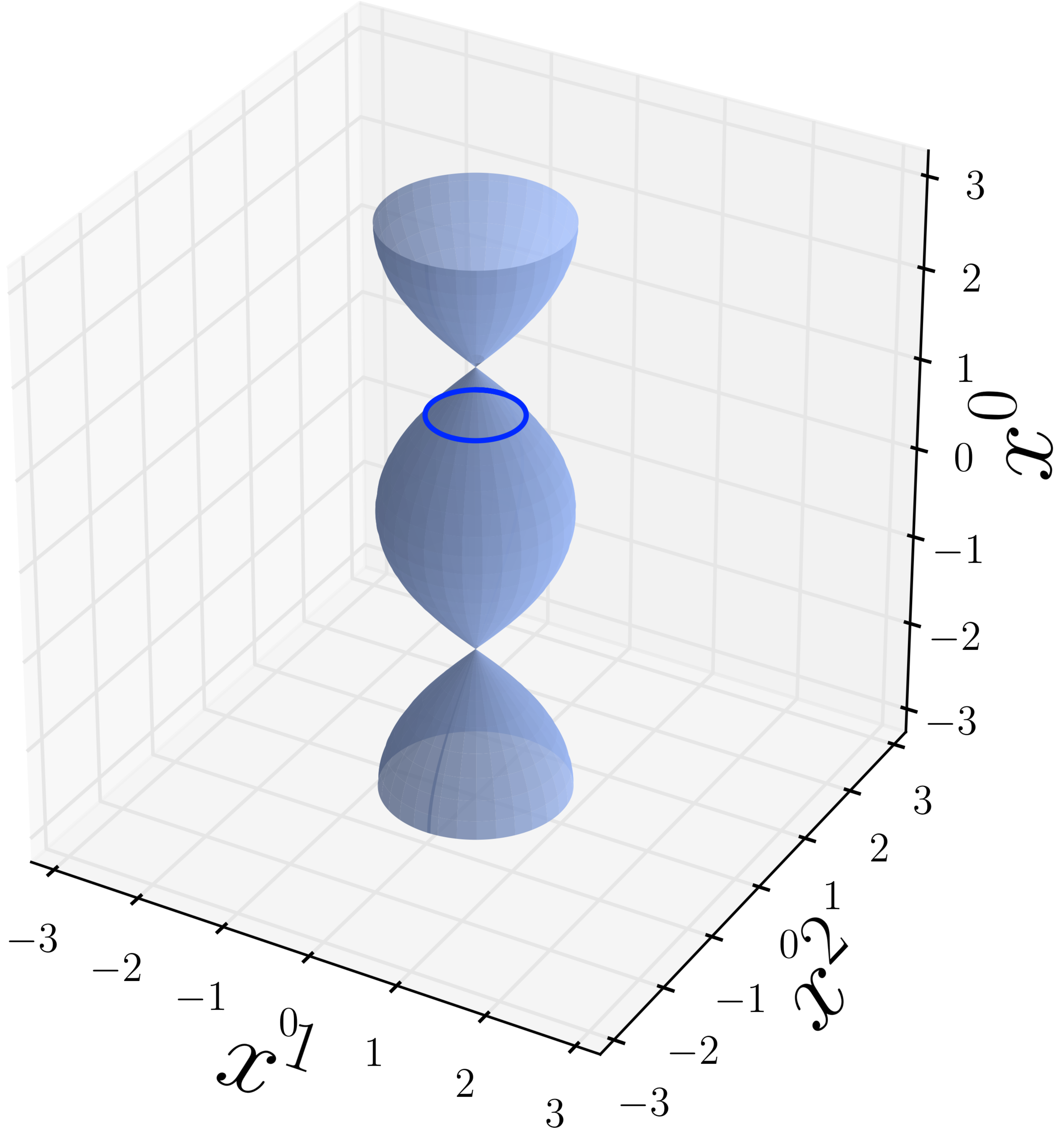

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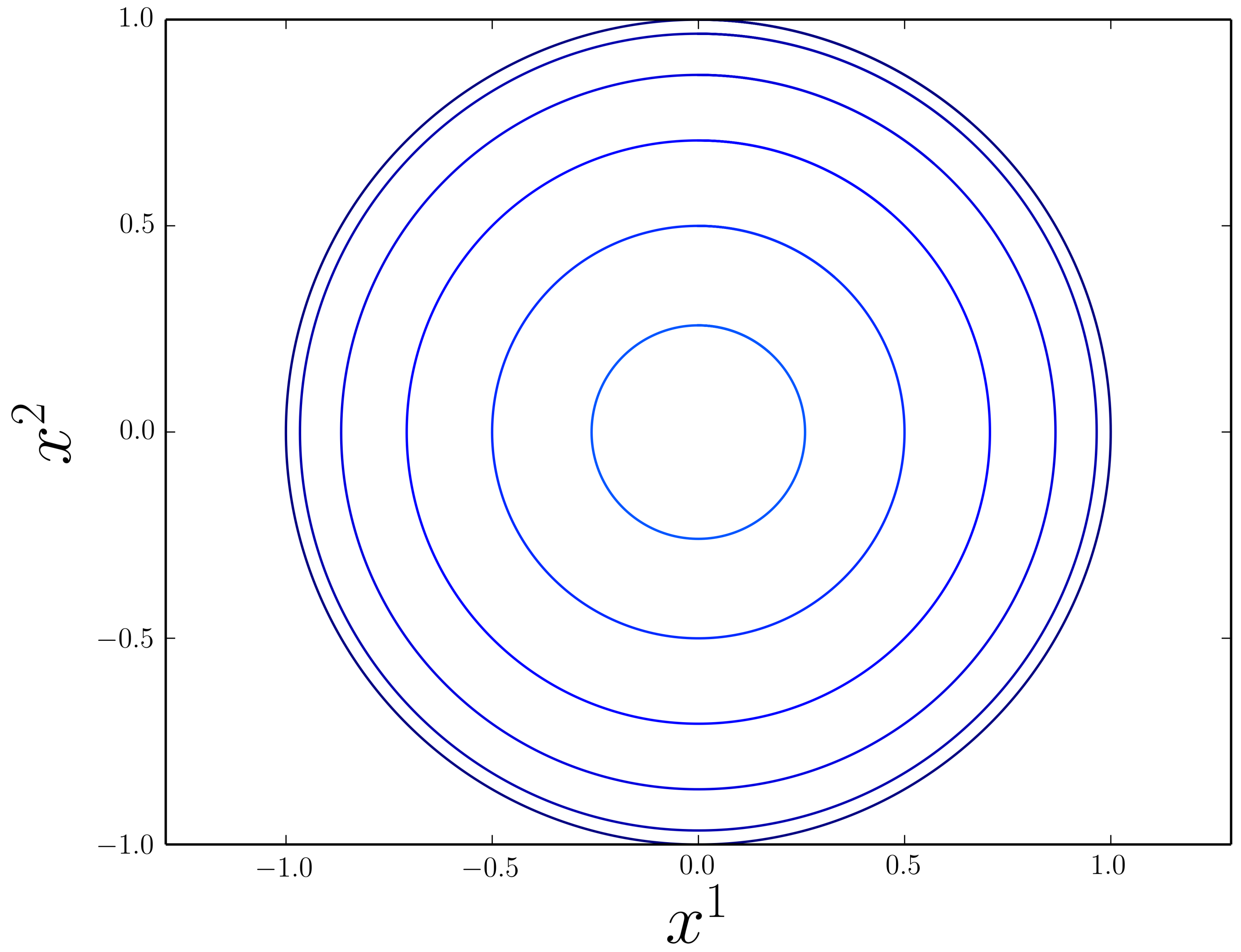

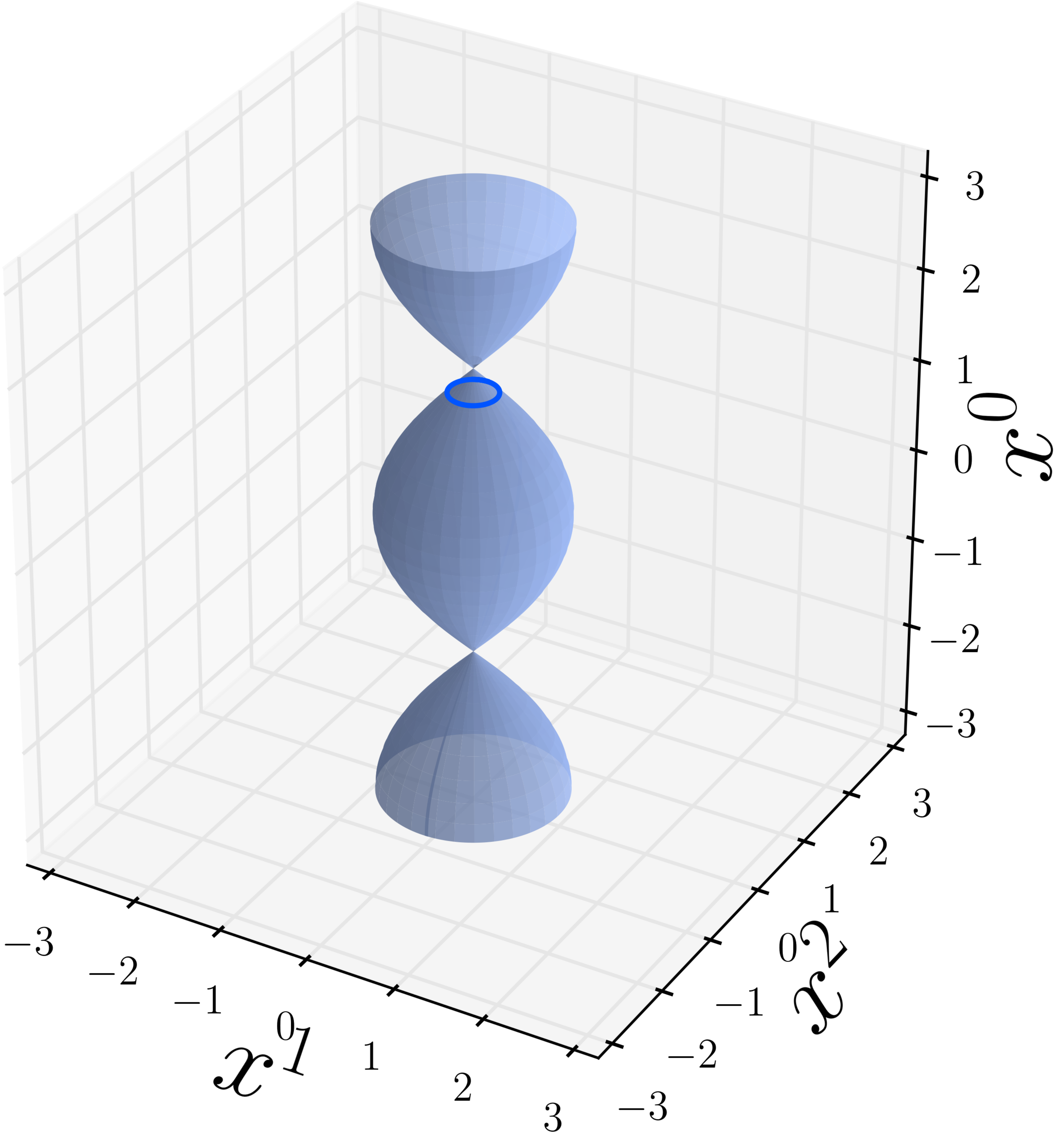

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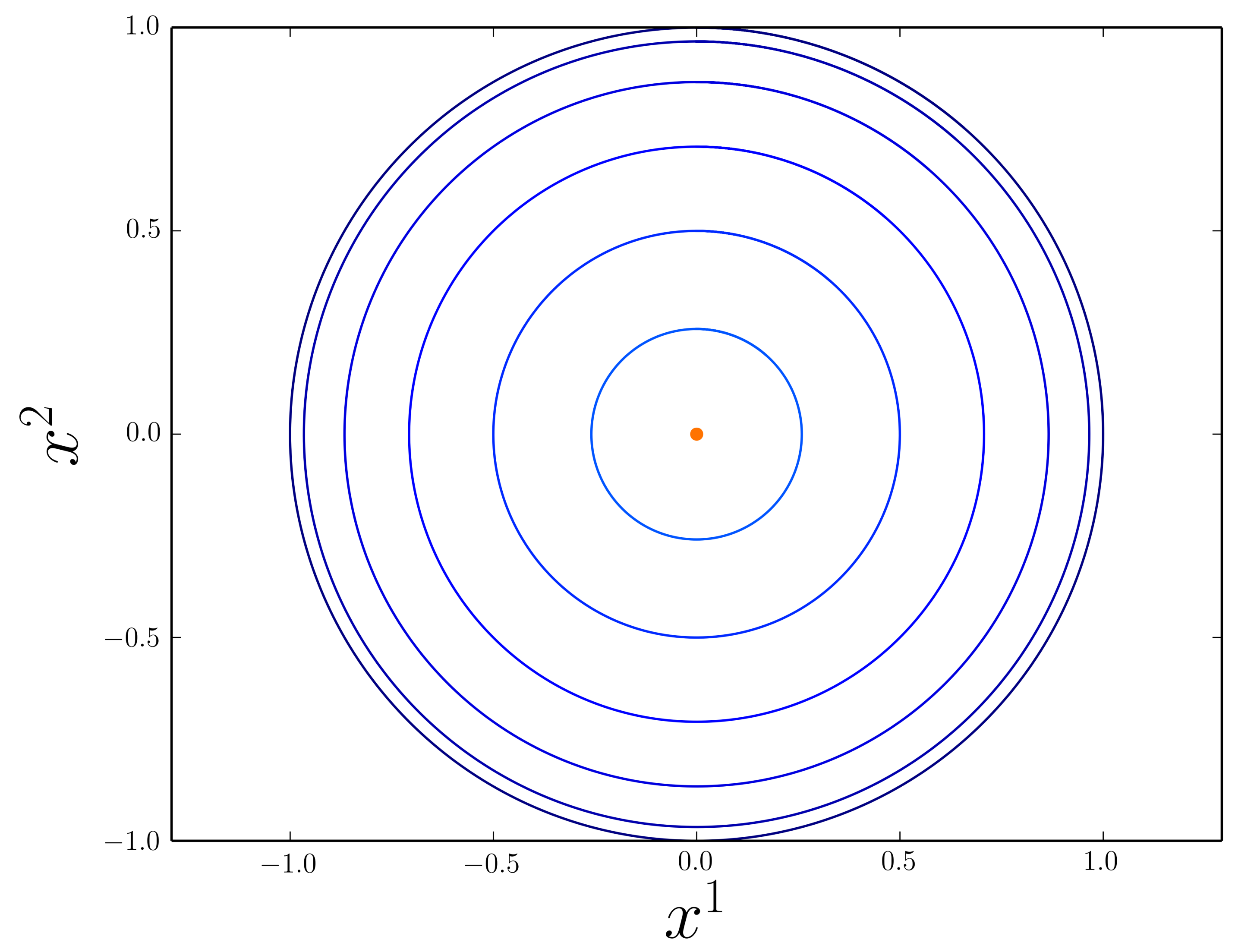
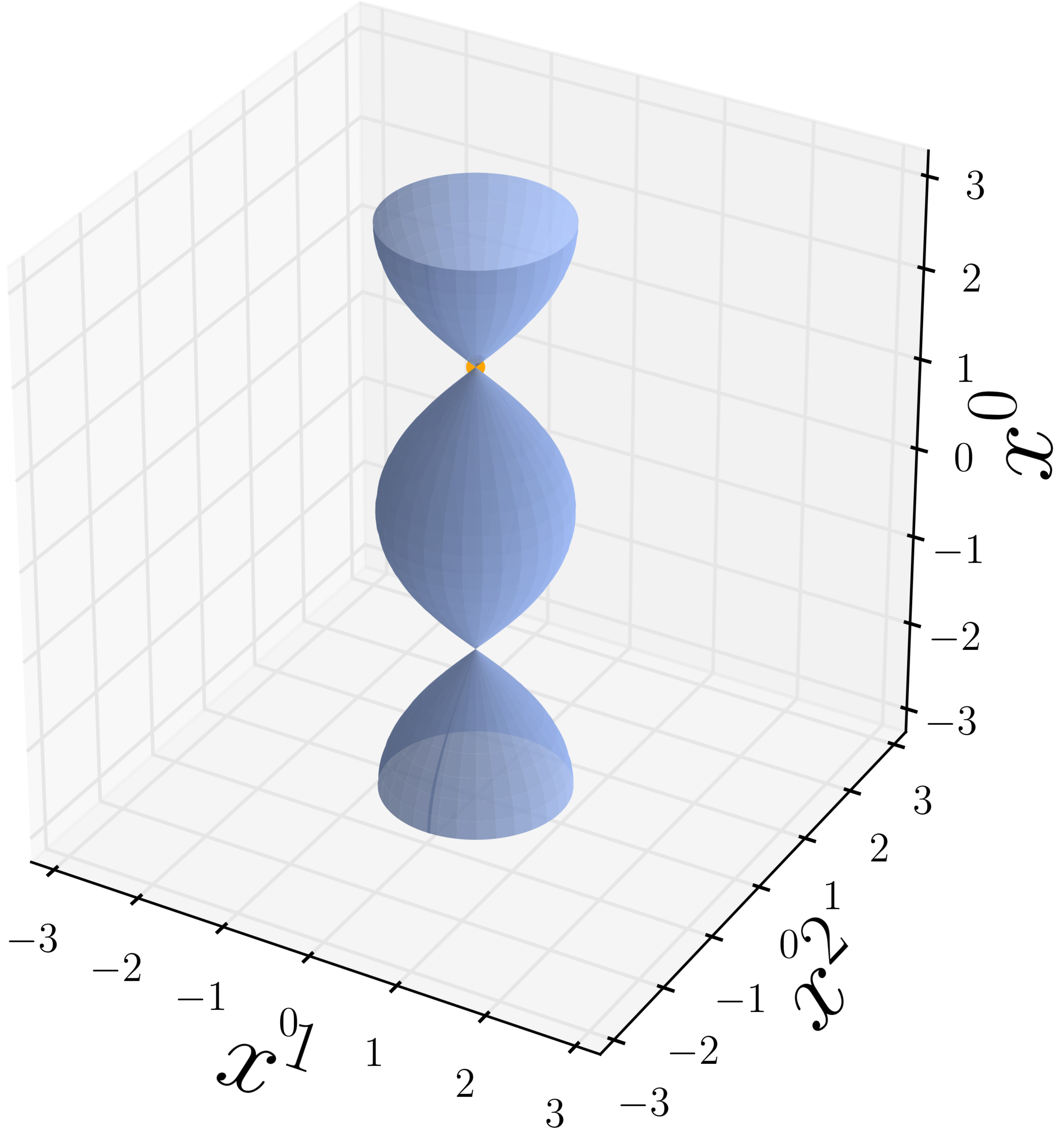
A flow of round circles which collapse to a point in finite time.





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Where did this example come from?

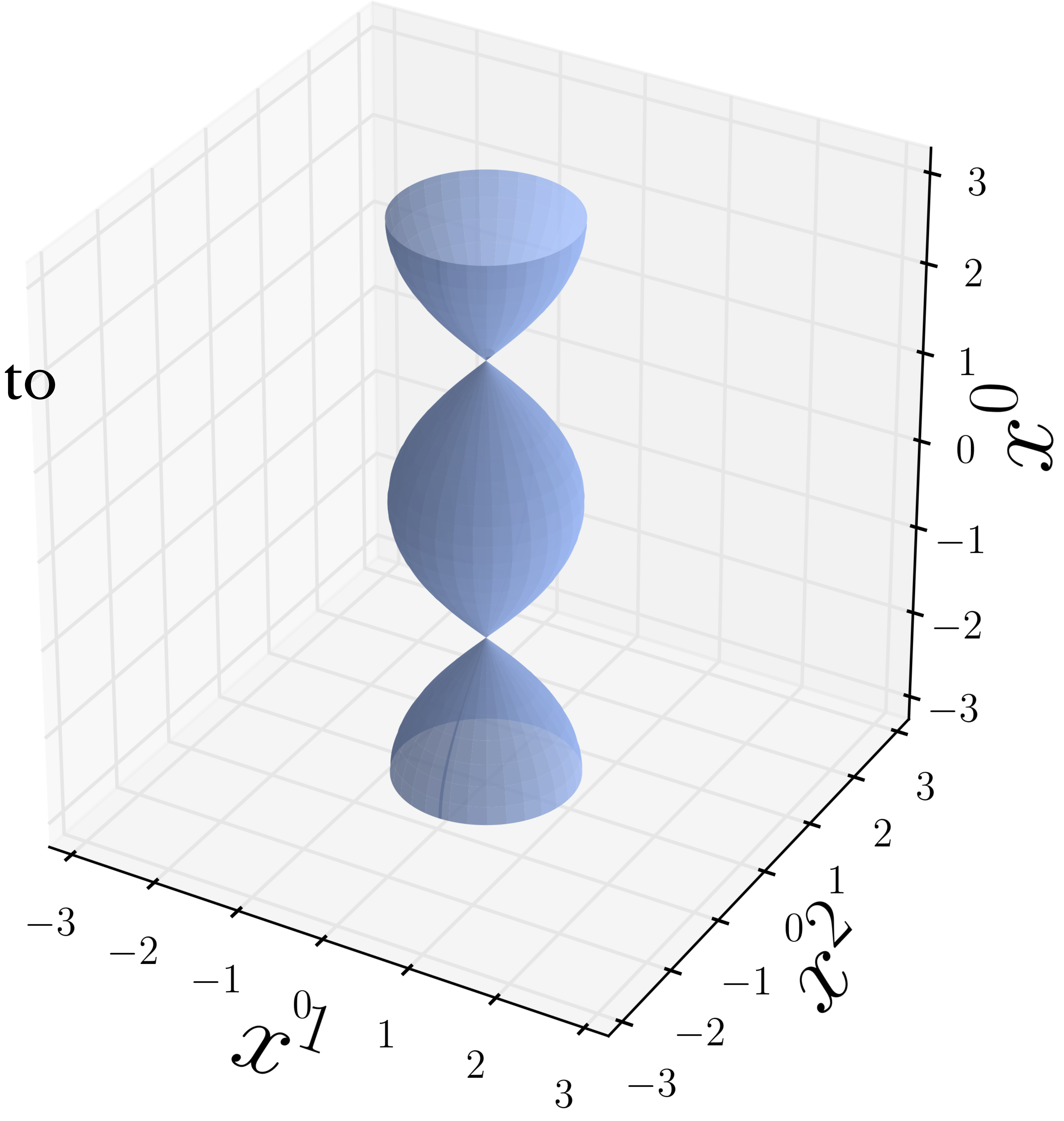
It is an example of the *method of isothermal gauge* (i.e. Weierstrass representation formula).

- This is a general trick for cooking up solutions to

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial t^2} = 0, \quad m\left(\frac{\partial\phi}{\partial t} \pm \frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t} \pm \frac{\partial\phi}{\partial s}\right) = 0$$

  which gives a TMS *provided ϕ is an immersion*.

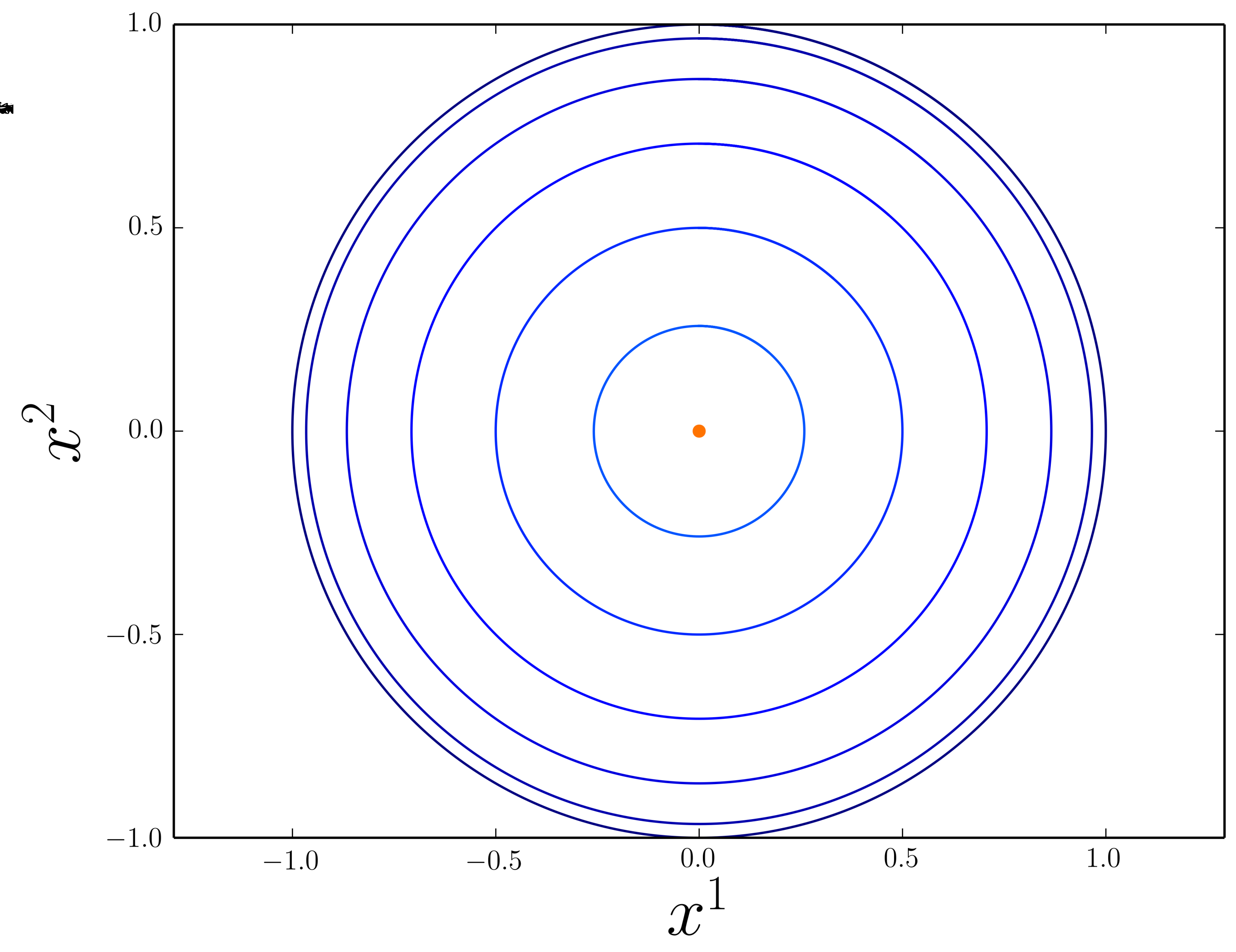
- It is a trick to cook up *singular* TMSs
- In general, it gives only local solutions to the Cauchy problem.

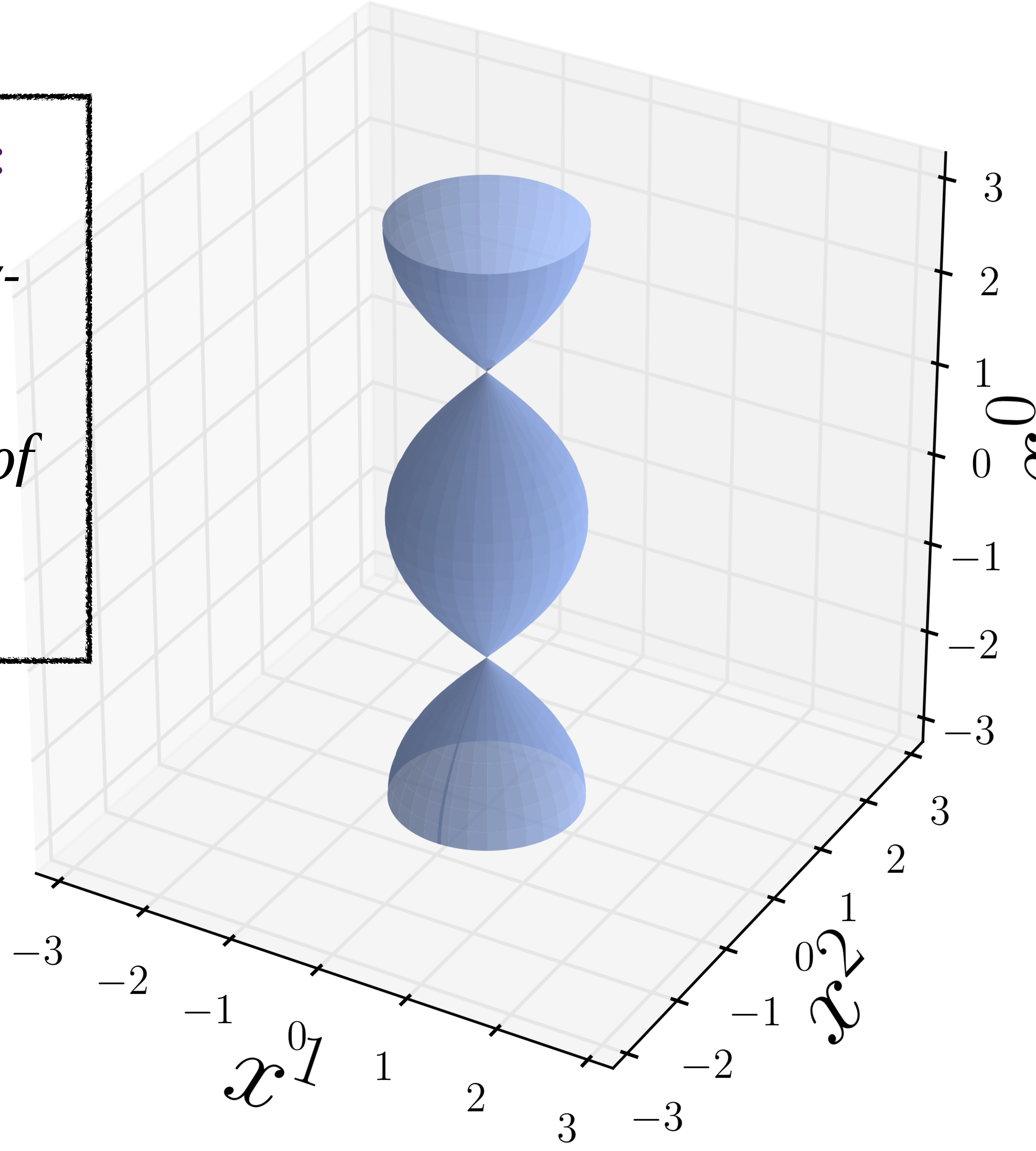


The shrinking circle is a special case of:

Theorem [Belletini, Hoppe, Novaga & Orlandi 2010]:

Let $C \subset \{x^0 = 0\}$ be a smooth, closed, convex, centrally-symmetric curve, and let $V = \partial_{x^0} = (1,0,0)$. Then the Cauchy evolution of (C, V) to a TMS consists of a family of smooth, closed, convex curves which shrink to a point singularity in finite time.





Note: This is not the *generic* singularity formation...

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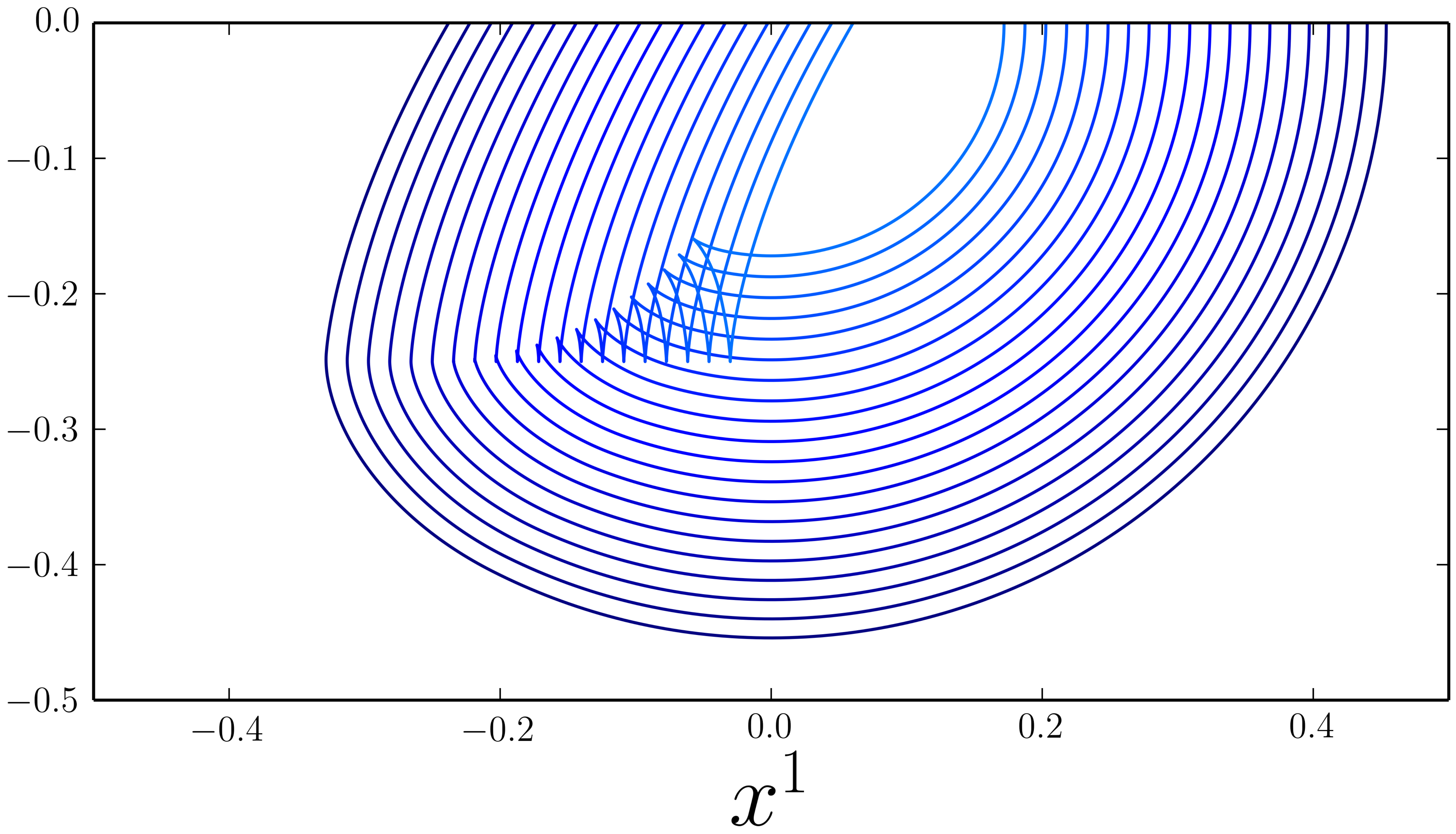
- *Generically*, for singular TMSs constructed by isothermal gauge, singularity formation is a *swallowtail*.
- At the onset of singularity, the limit curve is $C^{1,1/3}$. See [Eggers & Hoppe 2009] or [Nguyen & Tian 2013]).
- TMSs like to form singularities..



Theorem (Pron'ko, Razumov & Solov'ev 1983, Hoppe 1995, Nguyen & Tian 2013):

There exists no smooth proper timelike maximal immersion $\phi:\ S^1 \times \mathbb{R} \to \mathbb{R}^{1+2}$.

- I.e. No global solutions in the spatially-compact case.
- What about the spatially non-compact case?

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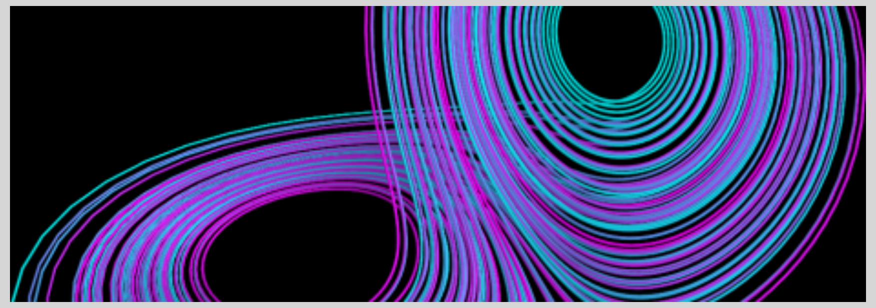

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Theorem 1 [P. 2019]:

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike maximal immersion. Then:

1. *ϕ is an embedding (i.e. no self-intersections).*
2. *For every compact subset $K \subseteq \mathrm{Im}(\phi)$, there is a timelike plane $P \subseteq \mathbb{R}^{1+2}$ such that K is a smooth graph over P.*

(An 'upside-down' Bernstein's theorem)

- **Corollary 1:** If $\Sigma \subseteq \mathbb{R}^{1+2}$ is a smooth properly-immersed TMS, and $N : \Sigma \to S^{1+1}$ is its (spacelike) unit normal, then $\mathrm{Im}(N)$ is contained in a closed hemi-hyperboloid.
- **Corollary 2:** If $C \subseteq \{t = 0\}$ is any self-intersecting curve and V any timelike vector field along C, then the Cauchy evolution of (C, V) to a TMS must form a finite-time singularity (in either the future or the past).

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Note 1: There exist many smooth properly immersed graphical TMSs. Take

$$\star \qquad \phi(s,t) = \left(t, \frac{1}{2}\left(c(s+t) + c(s-t)\right)\right)$$

where $c(s) = (x^1(s), u(x^1(s)))$ is a smooth proper graph parameterised by arclength.

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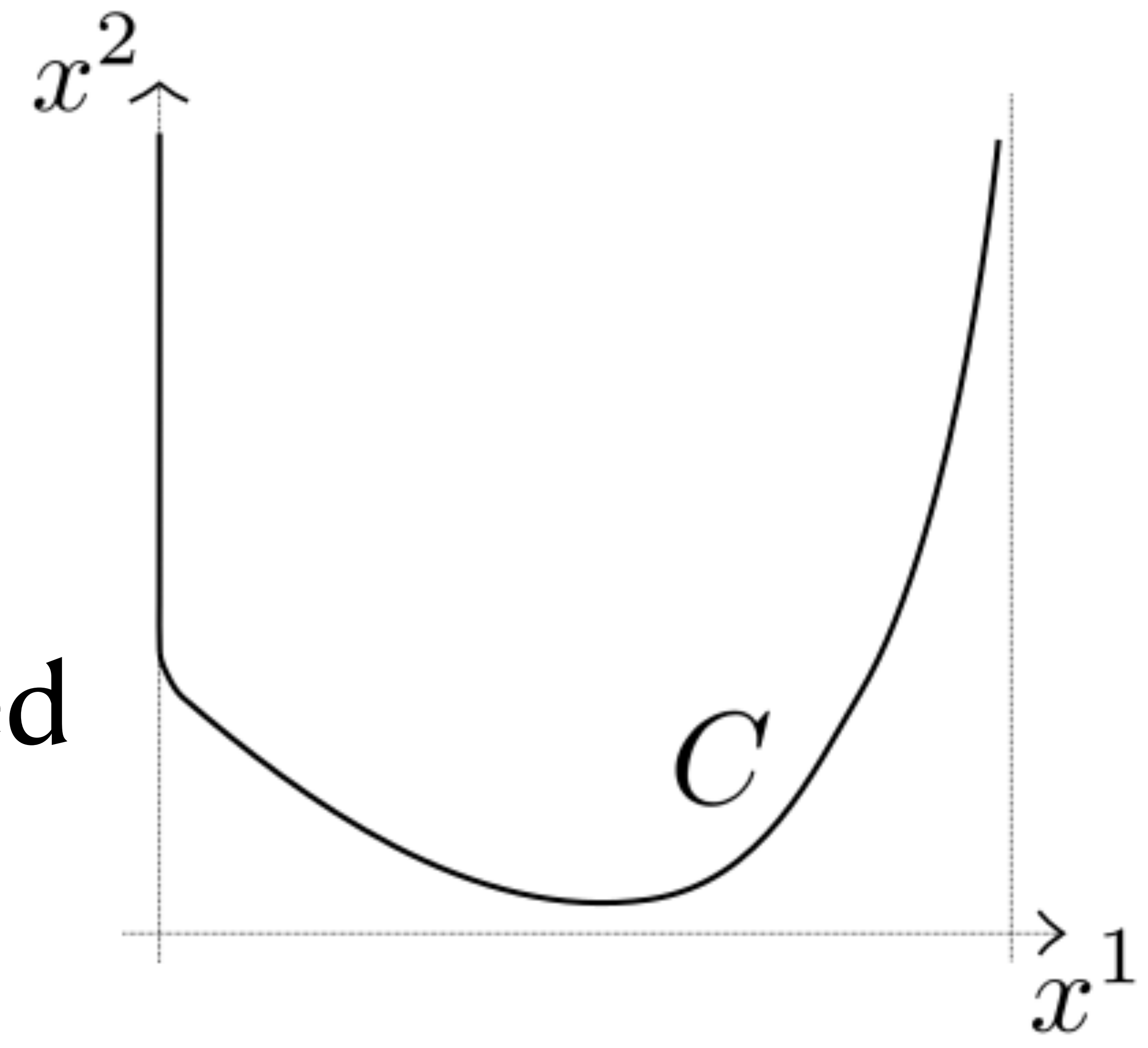
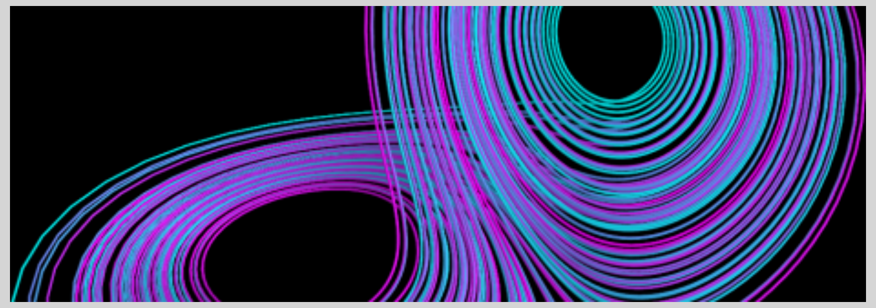


Note 2: This restriction cannot be relaxed. Take $\bigstar$ with c like this:



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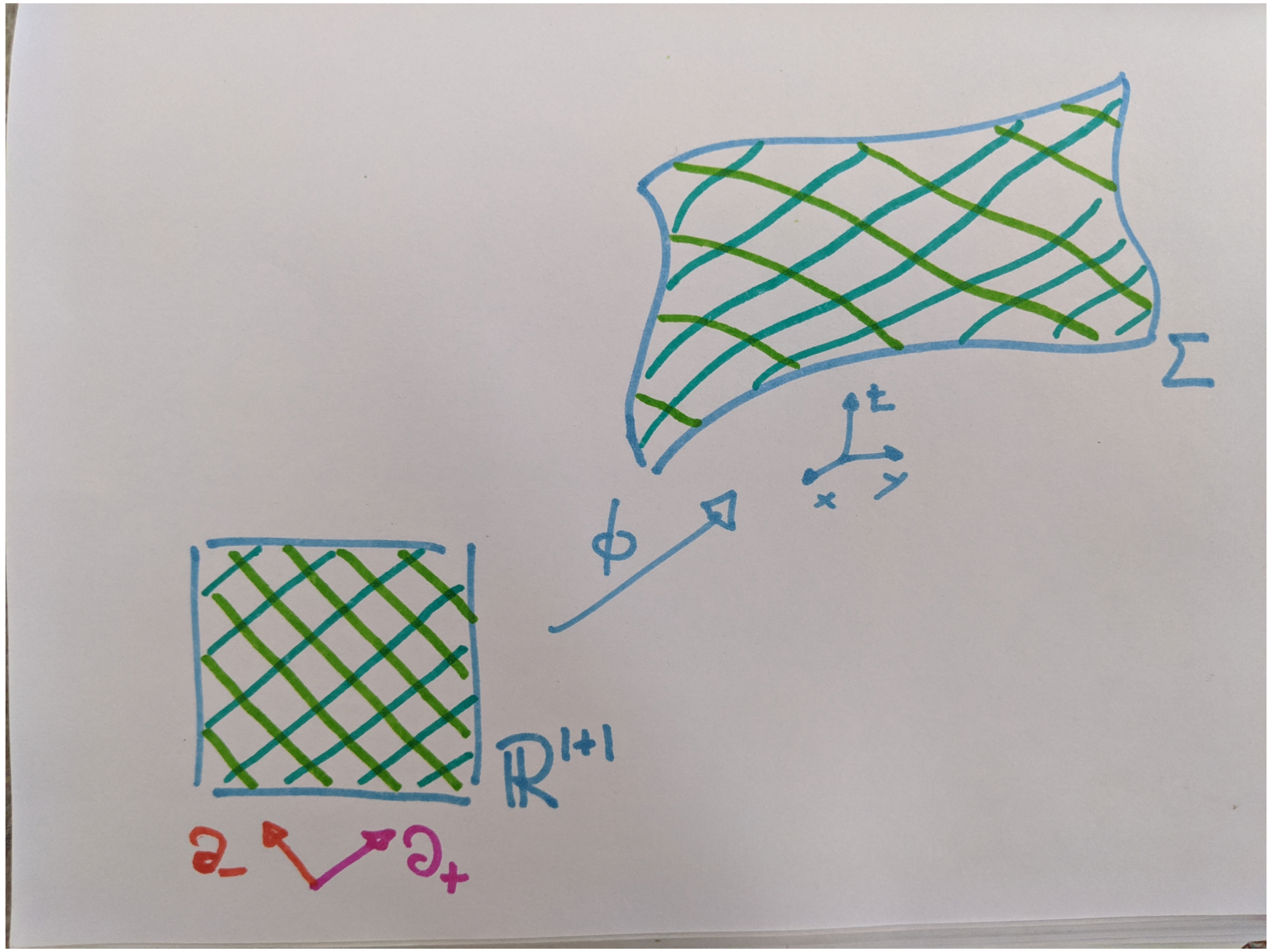
Sketch proof of Theorem 1: Let $\phi : \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike maximal immersion.

Step 1: Construct a coordinate change $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that, in the new coordinates (η_+, η_-),

$N_\pm = \dfrac{\partial \phi}{\partial \eta_\pm}$ are null (i.e global isothermal coords).

Existence of ψ proved by T. Milnor, 1985 (non-trivial!)



Sketch proof of Theorem 1: Let $\phi: \mathbb{R}^2 \to \mathbb{R}^{1+2}$ be a smooth proper timelike maximal immersion.

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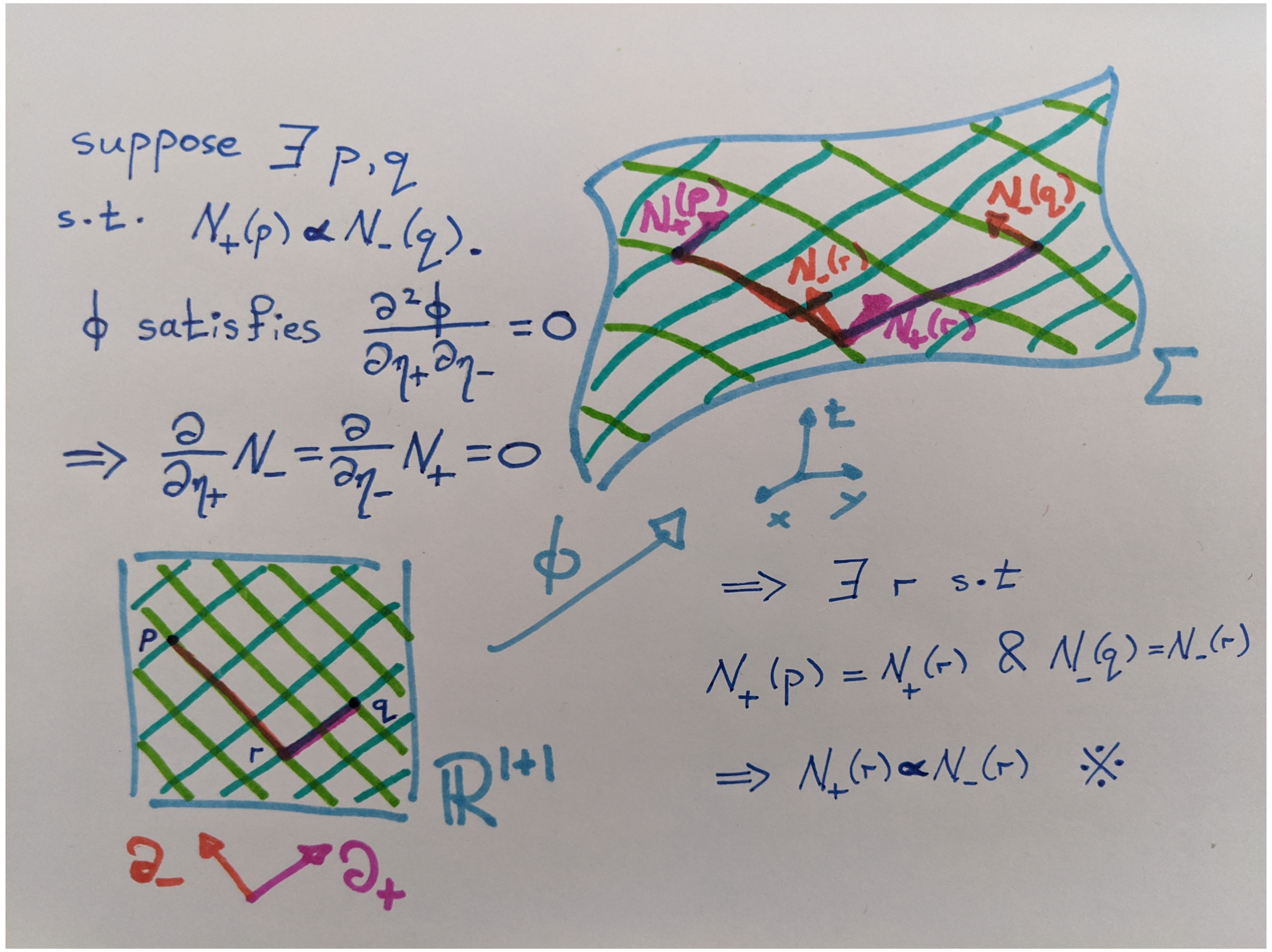

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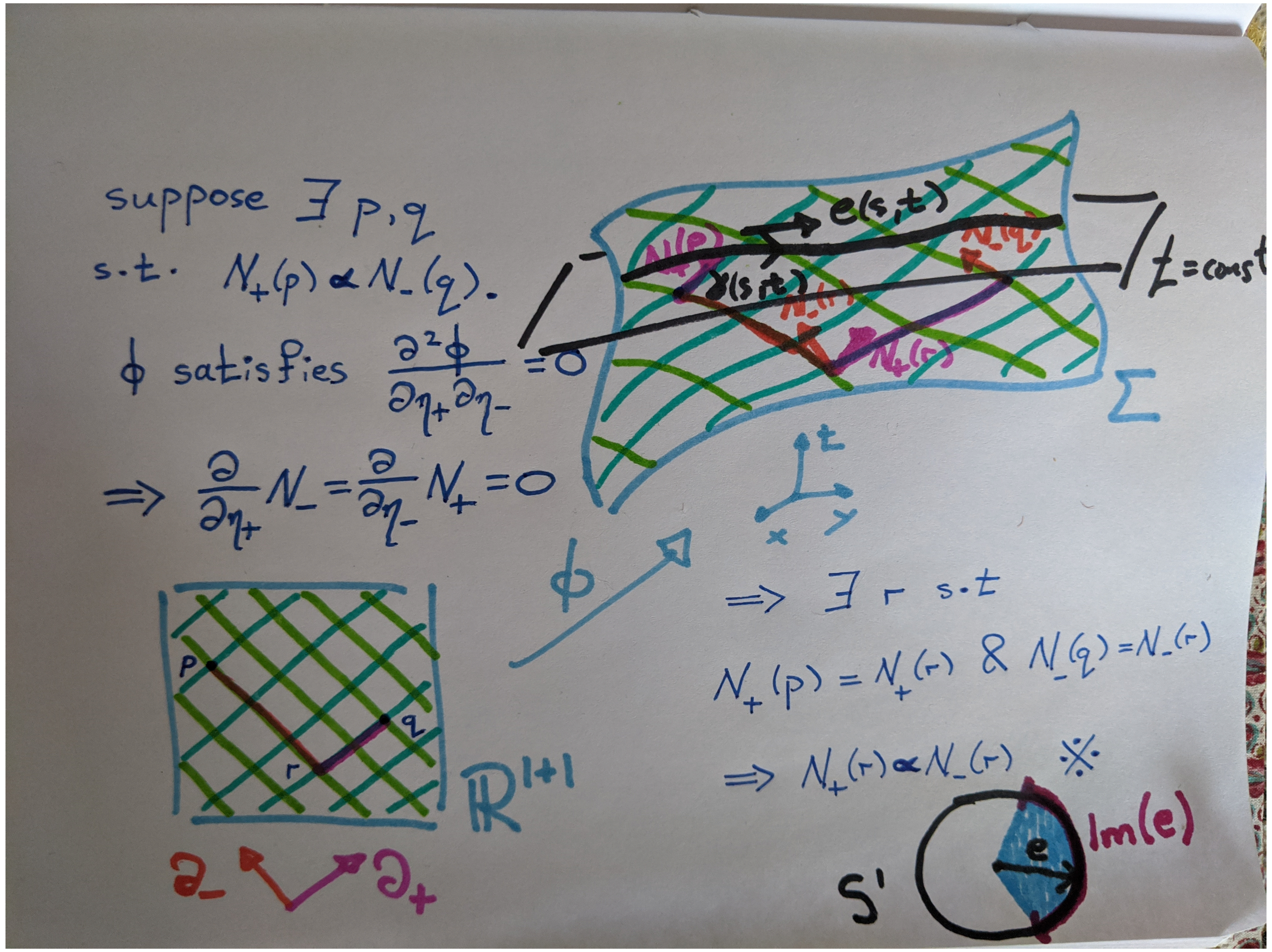

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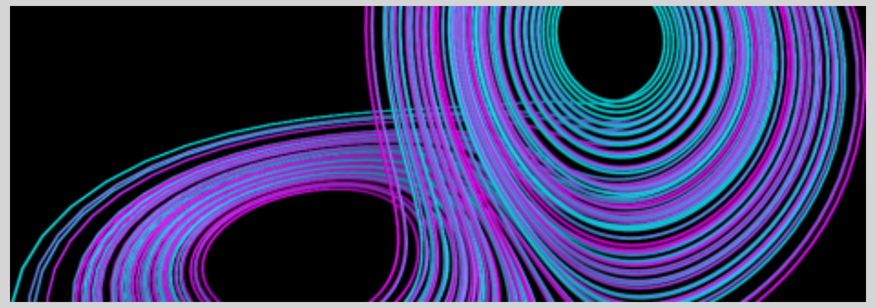
Step 2: Show that **Step 1** implies $N_+(p)$ and $N_-(q)$ are linearly independent for all $p, q \in \mathbb{R}^2$.

Step 3: Changing coordinates (WLOG) as $\phi(s,t) = (t, \gamma(s,t))$, write $e(s,t) := \dfrac{\partial_s \gamma(s,t)}{|\partial_s \gamma(s,t)|}$ for the spatial unit tangent along ϕ. Show that **Step 2** implies $\mathrm{Im}(e)$ is a strict subset of a closed semi-circle. Theorem 1 follows. ∎



- Recall that Theorem 1 implies:

  Corollary 2: If $C \subseteq \{t = 0\}$ is any self-intersecting curve and V any timelike vector field along C, then the Cauchy evolution of (C, V) to a TMS must form a finite-time singularity (in either the future or the past).

- But this does not reveal anything about the *nature* of singularity formation.

- A-priori, there are two things that could happen at a singularity:

  1. The surface fails to remain timelike.

  2. The surface fails to remain smooth.

- In fact (see e.g. [Jerrard, Novaga & Orlandi 2014] or [P. 2019]) case 1 always occurs.

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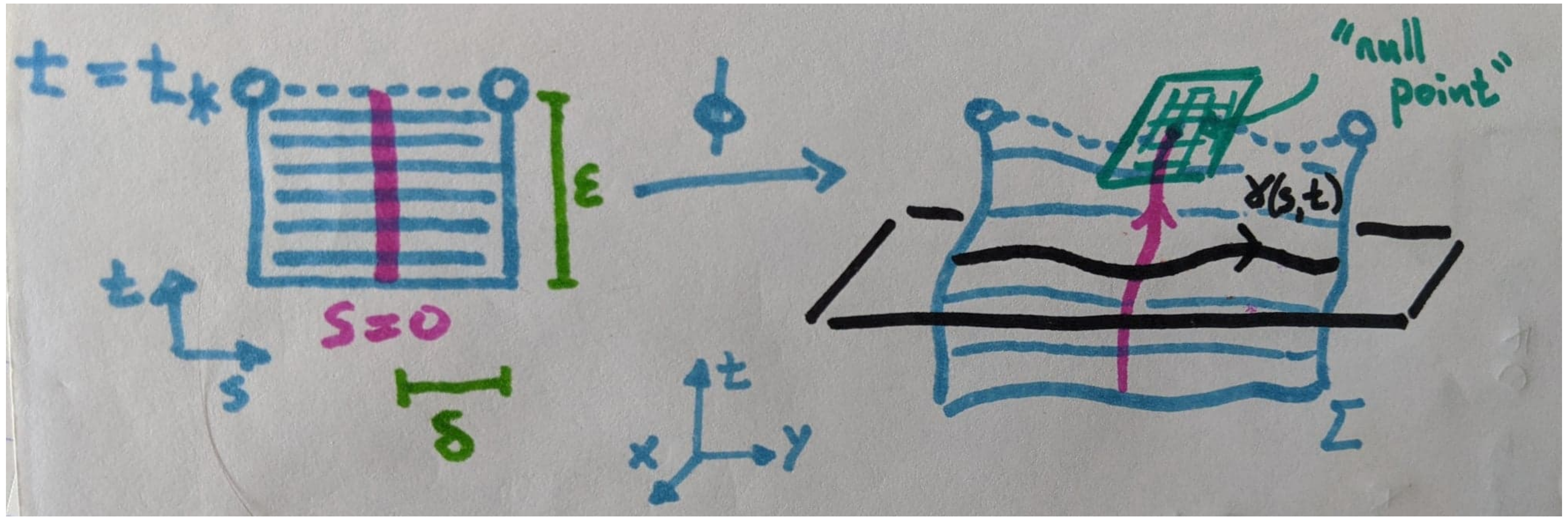
At the onset of a singularity ($t = t_*$), the evolution can be parameterised as:

$$\phi\colon [-\delta, \delta] \times [t_* - \varepsilon, t_*] \to \mathbb{R}^{1+2},$$

$$\phi(s,t) = (t, \gamma(s,t))$$

$$\langle \frac{\partial \gamma}{\partial t}(0,t), \frac{\partial \gamma}{\partial s}(0,t) \rangle = 0, \quad \lim_{t \uparrow t_*} \left| \frac{\partial \gamma}{\partial t}(0,t) \right| = 1$$

Here ϕ is a smooth timelike immersion on $[-\delta, \delta] \times [t_* - \varepsilon, t_*)$, but the spacelike unit normal blows up $\lim_{t \uparrow t_*} |N(0,t)| = \infty$. If ϕ is a C^1 immersion, then it is null at the point $(0,t_*)$.



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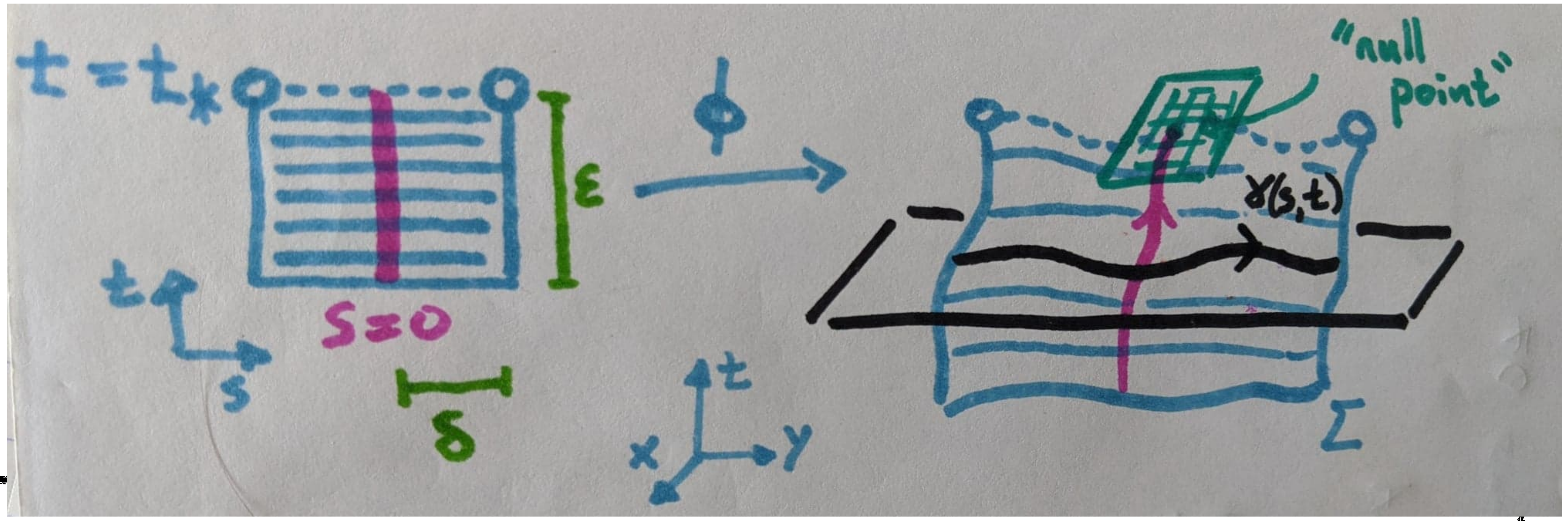

Theorem 2 [P. 2019]: Suppose ϕ is a timelike immersion as above, and suppose ϕ has bounded mean curvature scalar $|h(s,t)| \leq C$ for all $(s,t) \in [-\delta, \delta] \times [t_* - \varepsilon, t_*)$. Then

$$\int_{t_*-\varepsilon}^{t_*} |k(0,t)|\, dt = \infty$$

where $k(s,t)$ denotes the curvature of the (planar) curve $s \mapsto \gamma(s,t)$. In particular, ϕ is not C^2.

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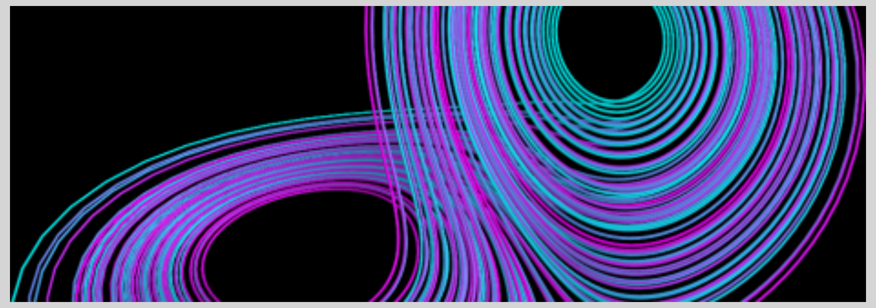

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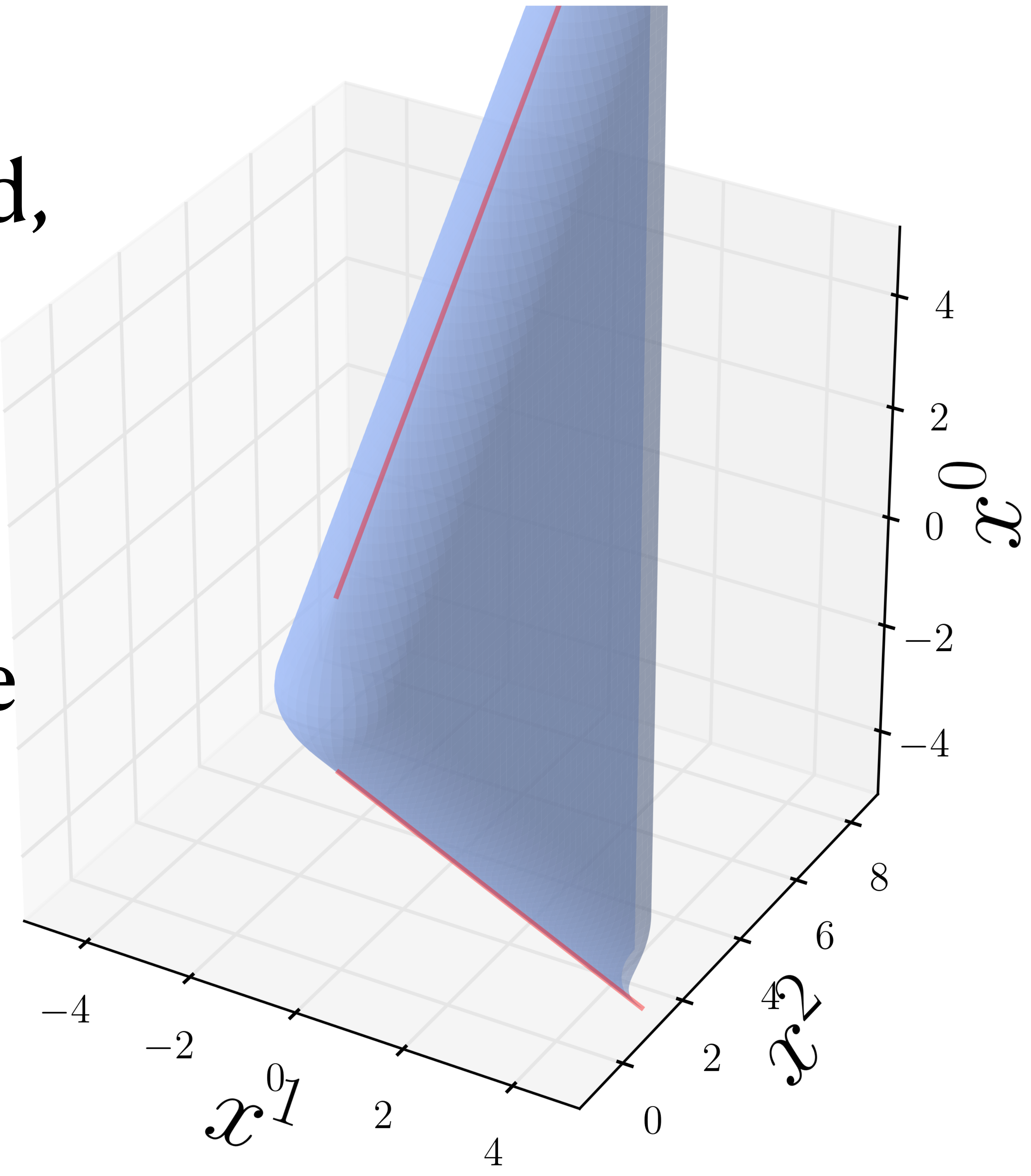
Wrapping up:

1. Every smooth properly immersed timelike maximal surface in $\mathbb{R}^{1+2}$ is embedded, and is a smooth graph over bounded subsets.

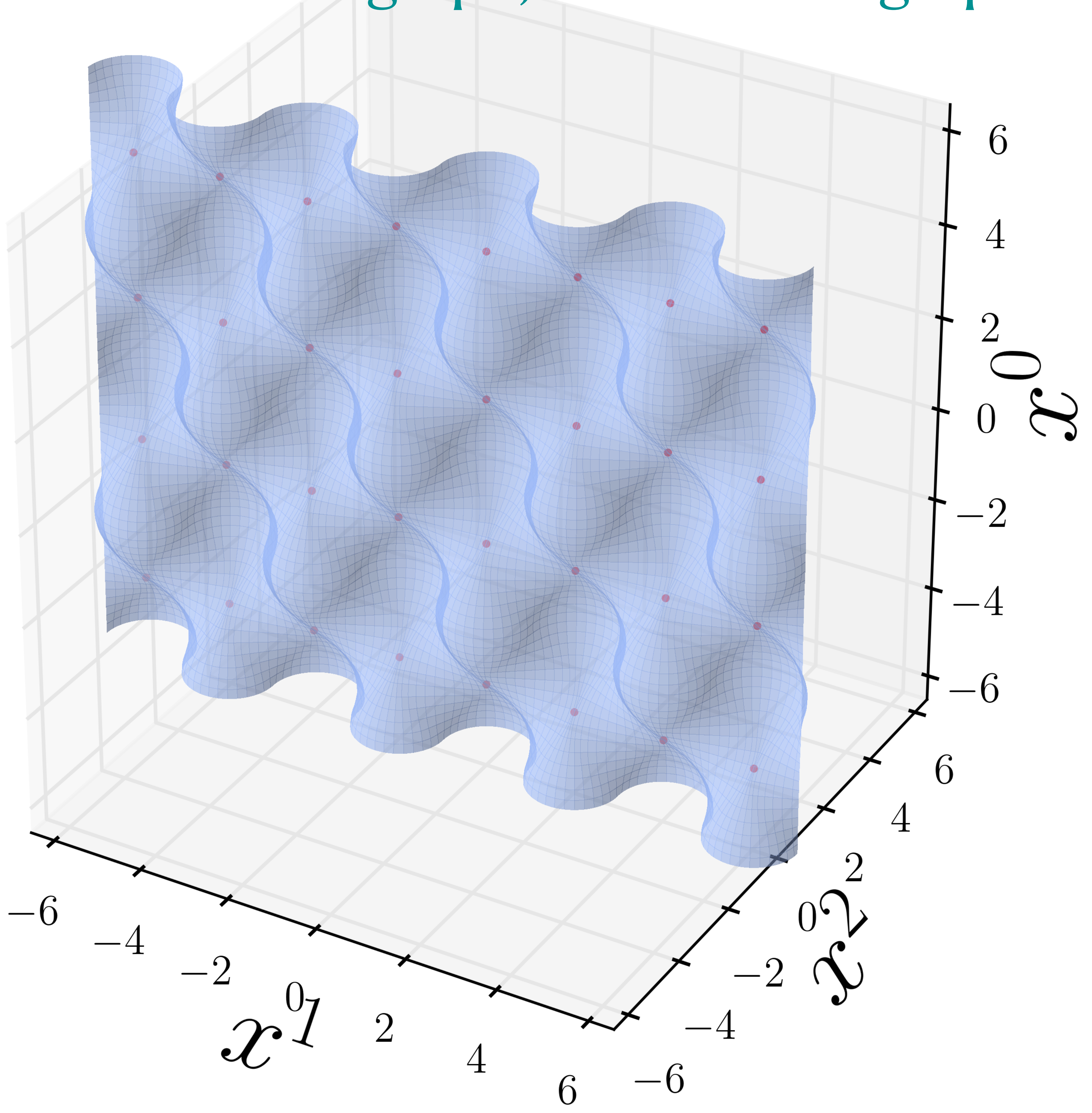
2. Singularity formation necessitates that the surface fails to be C^2.

3. But it might be C^1...

(i) A C^1 surface which is a smooth TMS away from a pair of null lines. It contains a compact subset which is not a graph.



(ii) A C^1 surface which is a smooth TMS away from a periodic lattice of null points. It contains a compact subset which is a C^0 graph, but not a C^1 graph.



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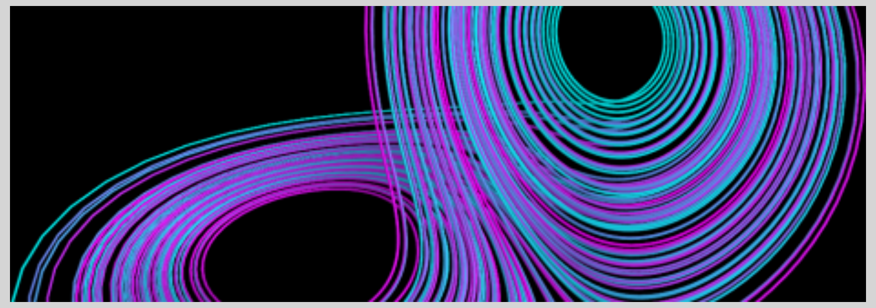

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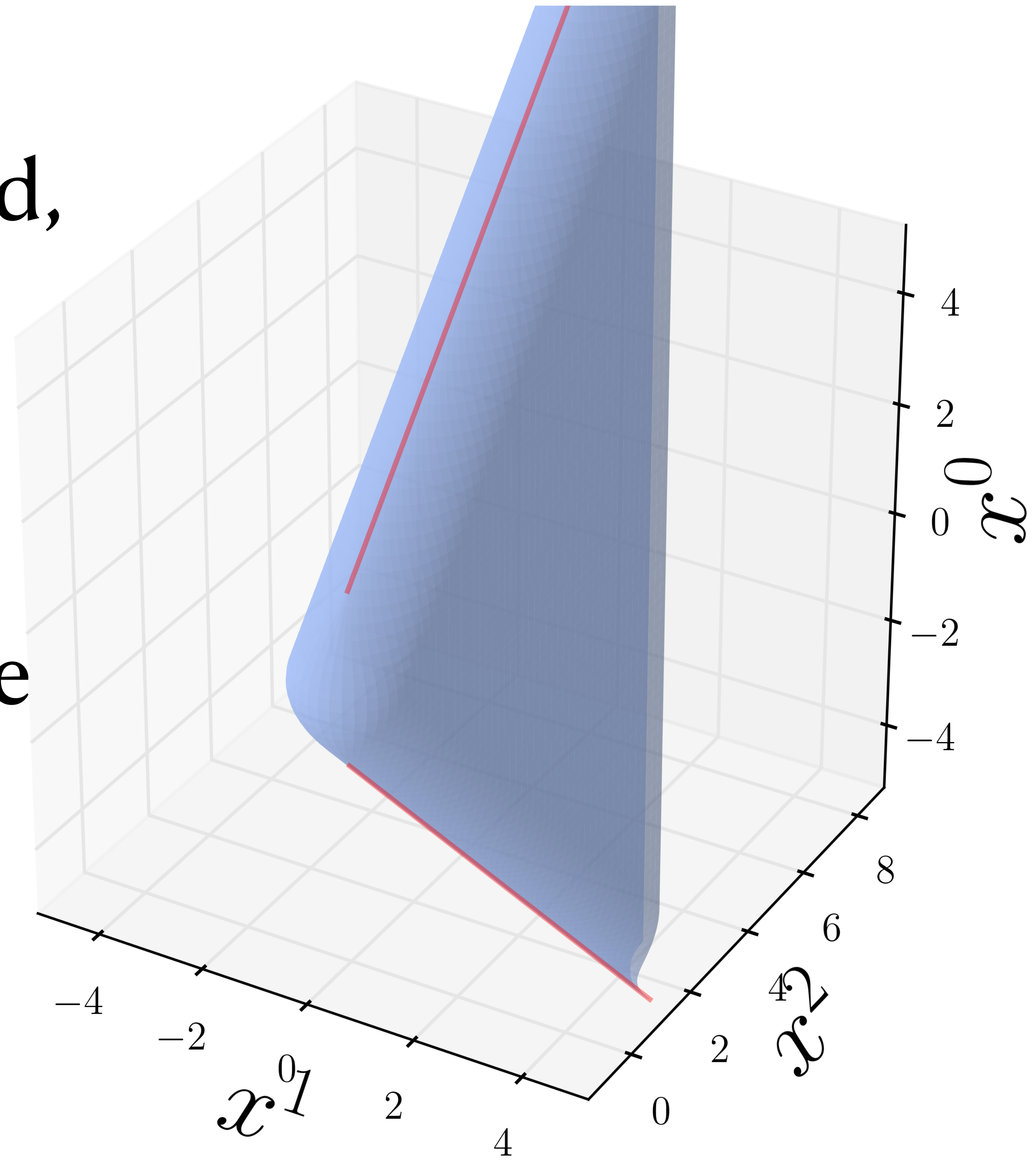

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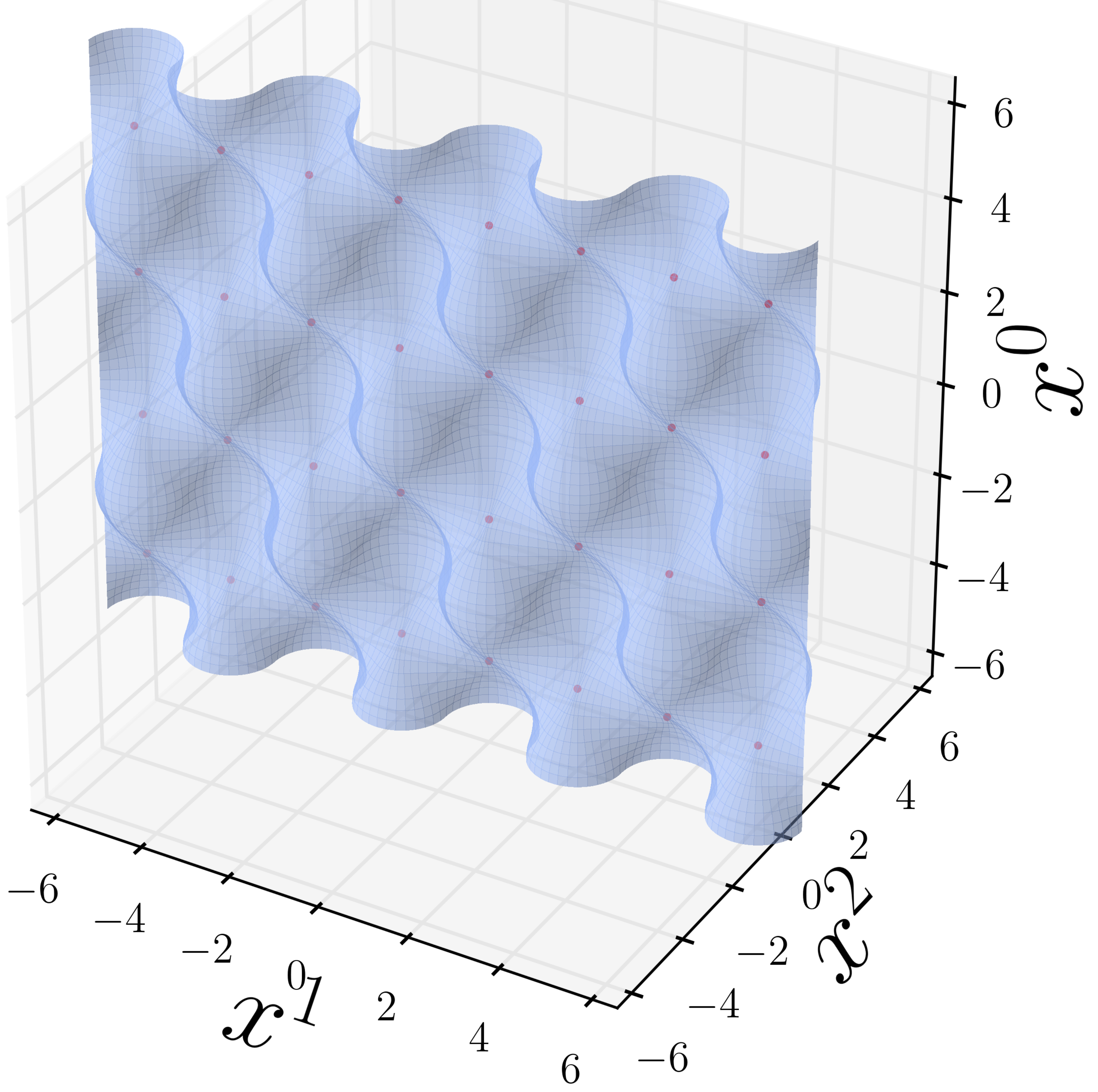

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